A 4-VALUED THEORY OF CLASSES AND INDIVIDUALS

Ross Thomas Brady

A Thesis Submitted for the Degree of PhD
at the
University of St Andrews

1971

A 4-VALUED THEORY

OF CLASSES

AND INDIVIDUALS.

I hereby declare that this thesis has been composed by myself, that the work of which it is a record has been done by myself, and that it has not been accepted in any previous application for any higher degree. This research concerning a 4-valued theory of classes and individuals was undertaken in October 1967, the date of my admission as a research student under Ordinance General No. 12 for the degree of Doctor of Philosophy (Ph.D.).

Ross T. Brady.

I hereby declare that the conditions of the Ordinance and Regulations for the degree of Doctor of Philosophy (Ph.D.) at the University of St. Andrews have been fulfilled by the candidate, Ross T. Brady.

Professor L. Goddard.

Acknowledgements.

I am deeply in debt to The Association of Commonwealth Universities and The British Council for their financial help without which the following thesis would not have been possible.

I am most grateful to Professor L. Goddard for his valuable guidance and the time and effort that he has devoted in supervising my work and in reading the manuscript of this thesis.

I am also very grateful to Professor R. Routley for his suggestion of the topic of the thesis and for his helpful suggestions made during his two visits to St. Andrews.

I would also like to thank Miss Robertson for duplicating the thesis from the stencils.

CONTENTS.

INTRODUCTION

1. Aims

The five main aims of the thesis are as follows:-

(1) To develop a 3-valued logic with values, truth, falsity and non-significance to handle certain types of sentences which cannot properly be dealt with using the classical 2-valued logic.

(2) To develop the theory of significance ranges, i.e. classes generated by predicates of the form, "It is significant that ϕx."

(3) To develop Goodman's theory of individuals in the same formal system as a theory of classes.

(4) To solve the problem of distinguishing individuals from the null class by using the above 3-valued significance logic.

(5) To avoid the paradoxes of class theory by adding an extra value to the above 3-valued logic.

2. The Need for a 3-valued Significance Logic

On an intuitive interpretation ~~In the usual semantics~~ of the 2-valued predicate calculus, the variable 'f' is taken to range over all predicates and the variable 'x' is taken to range over all subjects. According to the formation rules, 'fx' is a well-formed formula and hence 'fx', with a particular subject substituted for 'x' and a particular predicate substituted for 'f', is true or false. Take as predicate 'is in bed' and subject 'Saturday', where 'is in bed' has its obvious meaning and Saturday is a day of the week. Hence 'Saturday is in bed' is true or false according to ~~the usual semantics~~ this interpretation of the 2-valued predicate logic. This also applies to other grammatically well-formed sentences. For example, 'The number 3 is older than the number 4' (where the words have their usual meanings) is of the form $f(x,y)$, where 'f' stands for the relation 'is older than', 'x' stands for 'the number 3 and 'y' stands for 'the number 4'. So, 'The number 3 is older than the number 4' is also true or false according to the 2-valued logic. A problem now arises as to which value, truth or falsity, should be given to this type of sentence.

However, Ryle, in his paper Categories ((24)), wants to say that these sentences are neither true nor false. On page 68 he introduces the notion of sentence-factor. "Let us call any partial expression which can enter into sentences otherwise dissimilar a 'sentence-factor'." "Thus in the sentence, 'I am the man who wrote this paper', 'I', 'the man who', 'who wrote this paper', 'wrote this paper' are all sentence-factors." He goes on, on pages 69-70, "... the gap in a given sentence-frame can be completed by <u>some</u> but not by <u>any</u> alternative complements. But there are two sorts of 'can' here. 'So and so is in bed' grammatically requires for complements to the gap indicated by 'so and so' nouns, pronouns or substantival phrases such as descriptive phrases. So 'Saturday is in bed' breaks no rule of grammar. Yet the sentence is absurd. Consequently the possible complements must be not only of certain grammatical types, they must also express proposition-factors of certain logical types. The several factors in a non-absurd sentence are typically suited to each other; those in an absurd sentence, or some of them, are typically unsuitable to each other. To say that a given proposition-factor is of a certain category or type is to say that its expression could complete certain sentence-frames without absurdity." So Ryle wants to call sentences like "Saturday is in bed" absurd, rather than true or false, because "Saturday" does not express a proposition-factor of the right logical type to be coupled with "is in bed" (or, Saturday is not the sort of thing which can significantly be - or indeed, not be - in bed). Similarly, "the number 3" and "the number 4" do not express proposition-factors of the right logical type to be coupled with "is older than", and so "The number 3 is older than the number 4" is absurd, according to Ryle.

If what Ryle wants to say is so, then there is some sort of semantic inconsistency in this intuitive interpretation of the classical 2-valued logic, because it is being applied to Ryle's absurd sentences. There are three ways of avoiding this inconsistency: (i) One can accept that sentences like "Saturday is in bed" are true or false; (ii) one can restrict the substitution range of 'f' and 'x', as in type theory and other many-sorted systems; or (iii) one can accept that "Saturday is in bed" is absurd in Ryle's sense and make absurdity the third value in a 3-valued logic. I will discuss each of these alternatives in turn.

(i) Contrary to Ryle, if sentences like "Saturday is in bed" are true or false, so as to try to preserve the classical 2-valued logic, one must decide which one of these two values the sentences will take. Quine says that these sentences are necessarily false. He says, in ((18)) on page 229, "... the forms concerned would remain still quite under control if admitted, rather, like self-contradictions, as false (and false by meaning, if one likes)." If a value, truth or falsity, must be given to it, it seems that necessary falsehood is the only plausible choice because Saturday is not the sort of thing which can be in bed and so "Saturday is in bed" is not contingently true and not contingently false.

Now consider the sentence, "Saturday is not in bed". If the usual meaning is given to the negation, then "Saturday is not in bed" would be necessarily true. Goddard ((11)) has pointed out some difficulties about this assignment of values (see also Routley ((22)), pp. 180-182, Lambert ((15)), and Routley ((23))). The problem that arises here is, "Why is 'Saturday is in bed' false and its negation true rather than vice versa?" It seems that Saturday is not the sort of thing which cannot be in bed just as much as Saturday is not the sort of thing which can be in bed. As it stands above there is a correlation between negative sentences and truth and between positive sentences and falsity. As Goddard points out, "Neither in Mathematics nor elsewhere is there a general correlation between necessary truth (or indeed contingent truth) and negative sentences." He says that the correlation "rests on the ambiguity of denials". Because there is a psychological need to force a meaningful interpretation on to a sentence, "Saturday is not in bed" is interpreted as "Saturday is not the sort of thing which can be in bed". It is the latter sentence which is true, while the former is meaningless as is its positive form. This type of meaninglessness, which corresponds to Ryle's absurdity, I wish to call <u>non-significance</u>. Sentences which are not non-significant I wish to call <u>significant</u>. There is a case of ambiguity of denials in Drange's book ((4)) on page 24, where he argues:-

> "The theory of relativity is an abstract object.
>
> Abstract objects are not blue.
>
> The theory of relativity is not blue",

in support of the concluding sentence being true. As Goddard says, "either the second premiss is taken in the sense 'Abstract objects are not the kinds of things which could be blue' (which is the natural way to take it), in which case, the premisses are indeed true but the conclusion is 'The theory of relativity is not the kind of thing which could be blue'," which is of course true. "If the second premiss is taken as stated and not reinterpreted as a significance condition, then the whole argument begs the question by assuming that the second premiss is itself significant." Further discussion on the above quotation from ((4)) can be found in ((15)), pp. 82-83, and in ((23)).

There is a further objection to saying that "Saturday is in bed" is false and that its negation is true, and the like. Quoting from Goddard, "For suppose it is agreed that 'The Battle of Hastings likes tomato soup' is necessarily false and its negation 'The Battle of Hastings does not like tomato soup' is necessarily true. Presumably it would have to be said also that 'The Battle of Hastings dislikes tomato soup' is necessarily false, and its negation 'The Battle of Hastings does not dislike tomato soup' is necessarily true .... But given that the two negative sentences are true, then so is their conjunction 'The Battle of Hastings does not like tomato soup and does not dislike tomato soup'. In most ordinary contexts, however, to say that X does not like soup and does not dislike it, is to say that X is indifferent to soup, i.e. he can take it or leave it. So we are committed to the position that 'The Battle of Hastings is indifferent to tomato soup' is necessarily true. But this is a positive ascription of the same kind as the original sentence, 'The Battle of Hastings likes tomato soup', and ought therefore to be classified as necessarily false." More discussion on this point can be found in ((22)), pp, 181-182, ((15)), pp. 82-84, and ((23)).

This leads us to another suggestion that may be made to try to maintain the 2-valued predicate calculus. One may want to say that both "Saturday is in bed" and "Saturday is not in bed" are false. Goddard says, "In this case either (i) there is an immediate inconsistency; or (ii) the sentence and its negation are taken to be a sentence and its contrary, and then there is in at least some instances an implied inconsistency ...; or (iii) a new sense of "negation" is being introduced which holds for this special class of sentences alone. But this is enough to distinguish them. That is, we may now simply define non-significant sentences as those which satisfy the condition that both they and their "negations" are false." Thus a 3-valued logic is needed here to incorporate these "negations".

Thus there is no satisfactory assignment of values in 2-valued logic for Ryle's absurd sentences like "Saturday is in bed".

(ii) The second method for avoiding the inconsistencies is to restrict the substitution ranges of 'f' and 'x', so as to eliminate non-significant sentences from the formal theory. In this case, one would have to eliminate them in the formation rules so that no well-formed formula becomes non-significant for any substitution into its free variables. But, in order to include these semantic range conditions in the formation rules, one needs to presuppose a 3-valued theory of significance ranges.

An example of such range restrictions appears in Whitehead and Russell's book, Principia Mathematica ((34)), where the theory of types is used to produce them. $\psi(\chi)$, where ψ is a predicate of exactly one type higher than χ, is said to be significant (or meaningful). If ψ is not of exactly one type higher than χ, then $\psi(\chi)$ is said to be non-significant (or meaningless). Similarly for classes, if x is exactly one type lower than y, then $x \epsilon y$ is said to be significant (or meaningful), and if x is not exactly one type lower than y, then $x \epsilon y$ is said to be non-significant (or meaningless). They use the formation rules to exclude all non-significant sentences. Their theory of types suffers from the objection that they need a 3-valued logic, which would include variables ranging over all predicates and all classes (and individuals), to set up the type theory in the meta-language.

Even so, their notion of non-significance differs from Ryle's. For Ryle, $\psi(\chi)$ is non-significant iff χ is not the sort of thing which can have the property ψ. For example, "Saturday is in bed", which is significant according to the Whitehead-Russell theory of types, is non-significant according to Ryle's theory.

(<u>iii</u>) The remaining method of avoiding the inconsistency is to accept Ryle's theory and set up a 3-valued logic with values, truth, falsity and non-significance (Ryle's absurdity). Thus "Saturday is in bed" is non-significant, which is neither true nor false. The classical 2-valued logic must be rejected and a 3-valued logic must be substituted to cope with this type of sentence. One is forced to choose this third method as methods (<u>i</u>) and (<u>ii</u>) are both unsatisfactory. Henceforth, I will be assuming a 3-valued logic, the details of which I will be presenting in the first two chapters.

All through the discussion on the 3-valued significance logic, a philosophical problem was avoided. So far I have been referring to true, false and non-significant <u>sentences</u>. Normally, it would be said, propositions are true or false but not sentences. Also, one cannot have a non-significant proposition and so there seem to be two linguistic levels involved. Goddard (((10)) pp. 234-6) argues against the use of type-sentences:-

> "One might, for example, simply point to the fact that although it is true that if we so use the word 'rainbow' in a given context to refer to visual phenomena in the sky then particular sentence tokens of the type 'Rainbows eat flies' cannot be used to make significant statements, nevertheless we might also use the word in other contexts to refer to trout, and in this case tokens of the same type can be used to make significant statements. Hence, it seems, significance does depend crucially on the context of use, and we may express this by saying that any given token of the type 'Rainbows eat flies' can be used in three possible ways: to make a true statement, a false statement or an absurdity."

He goes on to say,

> "We most commonly do presuppose a standard context and judge significance in terms of it, but the significance of many simple sentences cannot be determined independently of their use even though we presuppose a standard sense for the words employed.
>
> Suppose somebody says 'Mary is happy'. Just because the word 'Mary' is used to name people, cows, ships and cyclones, so

we cannot, even given the standard sense of 'happy', determine the significance of the sentence independently of its use."

This supports the use of sentence-tokens instead of sentence-types, but these sentence-tokens must have some context of use. In their forthcoming book on the theory of significance, ((12)), Goddard and Routley will be examining this problem in full. Since, I believe, they have a satisfactory solution, I do not want to labour this point in my thesis. To give some idea, token sentences which are used to express true statements in a context are taken to be true, token sentences which are used to express false statements in a context are taken to be false, and token sentences which fail to express statements are taken to be either non-significant or incomplete. Non-significance occurs when there is a clash between the sense or reference of one part and the sense or reference of another part of a token-sentence. For example, the reference of "Saturday" to a day of the week clashes with the sense of "is in bed".

3. The Need for Significance Ranges

As already pointed out, the ranges of predicates need to be restricted if significance is to result. What we need to know is, "What are the sorts of things that can (or cannot) be in bed?" or "What are the sorts of things so that one thing can (or cannot) be older than another?"

It is convenient, therefore, to introduce the widest range of a predicate (or n-place relation) which will ensure significance. For the predicate "is red", the widest range consists of all extended things. Let this widest range of a predicate be called the <u>significance range of this predicate</u>. Similarly, for an n-place relation $R(x_1,-,x_n)$, the <u>significance range of R</u> is the class of ordered n-tuples $<x_1,-,x_n>$ such that $R(x_1,-,x_n)$ is significant. (This definition is due to Goddard and will appear in ((12)), the forthcoming book by Goddard and Routley.) Formally some kind of 3-valued Abstraction Axiom is needed. For example, $(Sy)(Ax)(x \in y \longleftrightarrow \phi(x))$, where there may be some restrictions on ϕ to avoid the class paradoxes, and where S is some sort of 3-valued existential quantifier, A is some sort of 3-valued universal quantifier and $\longleftrightarrow$ means "has the same value as". (This will be made more precise in later chapters.) Let the symbol Sp represent "It is significant that p". Then the significance range of a suitable ϕ is the class y such that $(Ax)(x \in y \longleftrightarrow S\phi(x))$. So these will be a special type of class with properties of their own and it will be interesting to examine them within the framework of a formal class theory, which will be done in Chapters IV, VII and VIII.

4. <u>The Need for a Theory of Individuals</u>

We now consider Goodman's theory of individuals, as developed in ((13))

and ((16)). This theory contains the relation $\leqslant$ ("is part of") between individuals. Instead of regarding a heap of stones, say, as a class of stones of which each stone is a member, we regard a heap of stones as a whole (or individual) which contains each stone as a part. In Goodman's theory, the universe of non-classes regarded as a whole (or individual) contains all individuals, which are, in fact, arbitrary parts of this universe. Goodman says (((16)) p. 45):-

> "The difference in the concepts lies in this: that to conceive a segment as a whole or individual offers no suggestion as to what these subdivisions, if any, must be, whereas to conceive a segment as a class imposes a definite scheme of subdivision - into subclasses and members."

In the above example four stones in the heap, considered as a whole (or "individual fusion" - ((16)), p. 47, I.03 - of the four stones) is just as much an individual as the whole heap of stones and as part of one stone in the heap. In fact, given a class of individuals, we can form the individual fusion of all the individuals of this class. Herein lies the essential difference between this theory and a theory of classes. Given some individuals, in the theory of individuals we "fuse" them together to form a new individual with each of these individuals as <u>parts</u>, whereas, in a theory of classes we form an abstract entity, a class with each of the above individuals as <u>members</u>.

So a theory of individuals can be used to formalise this notion of being a part in contrast to the notion of being a member. The formalisation of the theory of individuals also supplies us with a notion of individual identity which is defined in terms of '$\leqslant$'. The theory of individuals should be developed in the same formal system as a theory of classes because we want to form classes of individuals and not just classes of classes only. We want to include classes of stones, classes of people, classes of chairs, and the like.

5. Individuals and the Null Class

A problem that arises when one tries to introduce individuals into a theory of classes is that of distinguishing the null class from an individual, since both have no members. The class theory will contain an axiom of extensionality which will identify two classes or individuals if they have exactly the same members. The null class and an arbitrary individual will have no members and hence, by the axiom, be identical.

The difficulty is discussed in Quine's Set Theory and its Logic (((19)) pp. 29-32). One way out is to use separate variables for individuals and for classes or to introduce the primitive predicate "is an individual" into the system. Quine dismisses these as "unwelcome sacrifices of elegance" and says that happily these can be avoided. Quine instead suggests regarding $x \epsilon y$, where y is an individual, as $x=y$. This avoids the problem with the axiom of extensionality, because, if y and z are individuals $(Ax)(x\epsilon y \equiv x\epsilon z)$ is equivalent to $(Ax)(x=y \equiv x=z)$, i.e. $y=z$. Quine also shows that this implies that an individual is equal to its unit class and says that this does not affect the development of class theory as required for Mathematics. But if one takes a material object and forms its unit class, then, according to Quine, this material object would be equal to its unit class, an abstract entity, and this is unsatisfactory.

By taking $x\epsilon y$ as non-significant when y is an individual and using a 3-valued significance logic, one can avoid all the problems that have arisen in connection with distinguishing the null class from individuals. The predicate 'is an individual' can be defined in terms of the logic, i.e. $I(x) =_{df} \sim(Sy)S(y\epsilon x)$, i.e. $y\epsilon x$ is non-significant for all y, where the variables, x and y, range over classes and individuals.

The advantages of this over Quine's are obvious. No longer can it be said that there is an "unwelcome sacrifice of elegance". One can distinguish between individuals and their unit classes and avoid the identity of a material object with an abstract entity. The Axiom of Extensionality has to be restricted to classes using the predicate 'is a class' to restrict the general variables to class variables. The identity of individuals is established separately within the theory of individuals.

Just as '$\in$' is taken as a paradigm predicate used for generating classes by an abstraction axiom, membership of an individual can be taken as a paradigm case of non-significance for generating significance ranges. The rest of the classes can be obtained by adding arbitrary predicates, which does not affect the consistency nor the general features of the theory. Similarly with significance ranges, by the addition of arbitrary predicates, the significance ranges of these predicates can be formed.

6. The Class Paradoxes

In Cantor's naïve theory of classes, certain paradoxes came to light. Some of these paradoxes involved self-reference, where some form of self-membership is used to generate classes. Some paradoxes involved the size of the class of all classes. Cantor used an Abstraction Axiom of the form $(Sy)(Ax)(x\in y \equiv \phi(x))$, where ϕ is any predicate whatsoever. From this, the Russell Paradox can be derived as follows: Let $\phi(x)$ be $\sim(x\in x)$. Then $(Sy)(Ax)(x\in y \equiv \sim(x\in x))$. Hence, by letting this y be R, $(Ax)(x\in R \equiv \sim(x\in x))$ and $R\in R \equiv \sim(R\in R)$ which is a contradiction. Henceforth, I will call this R the Russell class. Another similar paradox is the Curry Paradox. Let $\phi(x)$ be $x\in x \supset f_z$. Then $(Sy)(Ax)$ $(x\in y \equiv . x\in x \supset f_z)$. Hence, by letting this y be C, $(Ax)(x\in C \equiv . x\in x \supset f_z)$ and

$C \epsilon C \equiv . C \epsilon C \supset fz$. Since $p \supset (p \supset q) \supset . p \supset q$, then $C \epsilon C \supset fz$. Hence $C \epsilon C$ and, by detachment, fz. Anything of the form fz is now provable and there is absolute inconsistency.

A paradox involving the size of the class of all classes is the Cantor Paradox. Cantor's Theorem states that the power class of a given class, x, has higher cardinality than that of x. It can be shown by using a reductio ad absurdum argument ending in a contradiction of the form $p \equiv \sim p$. But the class of all classes is the same class as its power class and hence must have the same cardinality. This is a contradiction which constitutes the Cantor Paradox. There are many class paradoxes of which the above three is just a sample. There have been many attempts in the literature to avoid these paradoxes. The usual method adopted has been to place restrictions on the predicates which generate classes according to an abstraction axiom.

Russell's theory of types, expounded in ((34)), only allows predicates of the form $\psi(x)$, where x is an individual variable (type 0) and ψ is a predicate of type 1, or predicates of the form $\psi(\chi)$, where ψ is a predicate of type exactly one higher than that of χ. The theory of orders also restricts the variables of quantification in the predicates. As previously mentioned, Russell appeals to non-significance in setting up his type theory and thus uses, in fact, a 3-valued system to avoid the class paradoxes.

In Quine's systems NF and ML, as discussed in Chapter 13, ((19)), stratification of the predicates is required. That is, whenever an expression $x\in y$ appears in the predicate there must be a uniform way of supplying indices so that the index of x is exactly one less than the index of y.

In the systems Z-F (as in Suppes, Axiomatic Set Theory ((30))) and NBG (as in Mendelson, Introduction to Mathematical Logic ((17))) an axiomatisation of the notion of set is achieved by allowing sets to be generated only from certain predicates. Also the ultimate classes of ML and the proper classes of NBG cannot be members.

There is another method of avoiding the paradoxes and that is to use a non-standard logic. The Russell Paradox has the form $p\equiv\sim p$ and the idea of a non-standard logic is to allow for this to be valid for some p, usually by rejecting the Law of Excluded Middle. The Cantor Paradox also arises out of a proposition of the form $p\equiv\sim p$, as explained above. Fitch in his book Symbolic Logic ((6)), has a consistent set theory using a natural deduction system which rejects general use of the Law of Excluded Middle. The use of the Lukasiewicz 3-valued logic is suggested in a number of places, e.g. in Fraenkel and Bar-Hillel's book Foundations of Set Theory ((7)) (pp. 193-4). Here it states that a form of the Curry Paradox is derivable. That is, (Sy)(Ax) $(x\in y \Longleftrightarrow .x\in x \longrightarrow .:x\in x \longrightarrow fz)$ and hence $A\in A \Longleftrightarrow .A\in A \longrightarrow .A\in A \longrightarrow fz$. In a 2-valued logic the Absorption Law, $p\supset(p\supset q)\supset.p\supset q$, allows the ordinary Curry Paradox to be derivable.

But, in the Lukasiewicz 3-valued logic, the law, $p \rightarrow (p \rightarrow .p \rightarrow q) \rightarrow .p \rightarrow .p \rightarrow q$, allows the proof of fz. Similarly, if an n-valued Lukasiewicz logic is used, then

$$\overbrace{(p \rightarrow .\cdots.p}^{n\ p's} \rightarrow q) \rightarrow \overbrace{.p \rightarrow .\cdots.p}^{(n-1)p's} \rightarrow q$$

also allows a form of the Curry Paradox to be derivable.

Wang (Survey of Mathematical Logic ((33)), p. 430) suggests using a 3-valued Lukasiewicz logic but allowing only ~, &, A to be used in forming predicates for the Abstraction Axiom. He also suggests using $\rightarrow$, but not to reiterate it, thus avoiding the Curry-type paradox. This presents an awkward problem since one has to keep a tag on each class as to whether $\rightarrow$ was used or not in its formation. In substituting for A in $(Sy)(Ax)(x \epsilon y \leftrightarrow .x \epsilon A \rightarrow x \epsilon B)$, A must not have been generated by a predicate with implication $\rightarrow$. A may have been obtained by successive substitution into many predicates and each one of these would have to be examined for $\rightarrow$. Anyway, the class C obtained, such that $(Ax)(x \epsilon C \leftrightarrow .x \epsilon A \rightarrow x \epsilon B)$, is a peculiar sort of construct from classes A and B. So Wang's suggestion is not very satisfactory. The only thing that can be said against his first suggestion is that, perhaps, the theory will not be strong enough to develop classical Mathematics. But, as will be shown in Chapter VI, the system NBG, which is strong enough to develop classical Mathematics, can be included in the 3-valued class theory as a 2-valued sub-theory.

Chang ((1)) tried to show the consistency of the Abstraction Axiom with unrestricted use of '$\longrightarrow$' in a Lukasiewicz infinitely-valued logic with only partial success. In his paper he showed that $(Sy)(Ax)(x \in y \longleftrightarrow \phi(x))$, where ϕ contains no free variable other than x, is consistent. ~~He and Fenstad ((5)) have improved on this by allowing ϕ to contain bound variables of a specified sort~~. Earlier Skolem ((25)) had already shown that $(Sy)(Ax)(x \in y \longleftrightarrow \phi(x, u_1, \text{--}, u_k))$, where ϕ is quantifier-free, is consistent. Chang((1)) and Fenstad((5)) have improved on this by allowing ϕ to contain bound variables of a specified sort. Even though these partial results have been obtained the consistency of the unrestricted Abstraction Axiom remains an open question. But, as said earlier, the admission of $\longrightarrow$ into the Abstraction Axiom leads to peculiar classes being formed.

There has also been some work done by Skolem ((26)) on the consistency of the Abstraction Axiom in 3-valued logic. He has shown in his paper that $(Sy)(Ax)(x \in y \longleftrightarrow \phi(x, u_1, \text{--}, u_k))$, where ϕ is quantifier-free and contains connectives $\sim$ and & only, together with the Axiom of Extensionality, is consistent. I will show in Chapter V that $(Sy)(Ax)(x \in y \longleftrightarrow \phi(x, u_1, \text{-}, u_k))$, where ϕ contains connectives $\sim$, &, and quantifier A only, together with the Axiom of Extensionality, is consistent (relative to Z-F).

A question now arises as to what connectives and quantifiers of 3-valued logic can be added to $\sim$, & and A in the Abstraction Axiom, while still maintaining consistency. Skolem shows ((26)) that relative quantification cannot be added as well as absolute quantification. It is also inconsistent to add $\supset$, represented as:-

$\supset$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	1
0	1	1	1

Since $p \supset q \Longleftrightarrow .p \longrightarrow .p \longrightarrow q$, a version of the Curry Paradox results. The addition of T, represented as:-

T	
1	1
$\frac{1}{2}$	0
0	0

also leads to inconsistency since $(Sy)(Ax)(x \epsilon y \Longleftrightarrow T{\sim}(x \epsilon x))$ implies $A \epsilon A \Longleftrightarrow T{\sim}(A \epsilon A)$, for some constant A, and no value ($1, \frac{1}{2}$ or 0) can be consistently given to $A \epsilon A$. It is also inconsistent to add $\Longleftrightarrow$, represented as:-

$\Longleftrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$
0	0	$\frac{1}{2}$	1

since $(Sy)(Ax)(x \epsilon y \longleftrightarrow .x \epsilon x \Longleftrightarrow R \epsilon R)$, where R is the Russell class, leads to $A \epsilon A \Longleftrightarrow .A \epsilon A \longrightarrow R \epsilon R$, for some constant A, in which A cannot be consistently given a value. So it appears that none of the more usual connectives or quantifiers can be added to $\sim$, &, A so as to preserve consistency of the Abstraction Axiom. I will defer an examination of the merits of this 3-valued approach until the conclusion because then I will be in a better position to assess them.

I will briefly describe how the 3-valued approach avoids some of the class paradoxes. In the derivation of the Russell Paradox, one ends up with $R \in R \leftrightarrow \sim(R \in R)$. This is still obtained using the 3-valued logic. $R \in R$ cannot have the value 1 or 0 and so it is given the value $\frac{1}{2}$. In this case, by the 3-valued logic, $R \in R \leftrightarrow \sim(R \in R)$ takes the value 1, and the Russell Paradox is avoided.

In the 3-valued logic one can still derive $C \in C \leftrightarrow .\sim C \in C \vee .\sim C \in C \vee fz$, which is the closest one can get to the Curry Paradox because of the restrictions on the connectives. In this case, if fz takes the value 1, 0 or $\frac{1}{2}$ then $C \in C$ takes the value 1, $\frac{1}{2}$ or $\frac{1}{2}$, respectively. Thus the Curry Paradox is avoided.

In the proof of Cantor's Theorem in 3-valued logic the reductio proof fails because $p \equiv \sim p$ is no longer a contradiction. In fact, the power class of the class of all classes is the class of all classes itself and hence there can be a greatest cardinal number, viz. that of the class of all classes. This now invites the question, "To what classes does Cantor's Theorem apply?"

To answer this, I distinguish between classes and special classes, where every special class is a class but not every class a special class. The notion of special class is such that the classical 2-valued logic (or a 3-valued significance logic) applies to them, while the notion of class is such that the Axioms of Abstraction and

Extensionality hold. The special class theory I will adopt is NBG because it has fewer difficulties than Quine's systems and is more general than Z-F. This theory is strong enough to develop classical Mathematics and can, as said earlier, be contained in the 3-valued theory as a sub-theory.

At this point, I want to raise the question, "Can the value non-significance be the same as the value $\frac{1}{2}$ of the Lukasiewicz logic used above to avoid the class paradoxes?" Certainly this was the case with Russell, because, in his theory of types, the so-called paradoxical statements (like $R\epsilon R$) are non-significant since they violate type rules. However this is not the case when the two values are used as described above. For consider the two examples, $x\epsilon a$, where a is an individual, and $R\epsilon R$, where R is the Russell class. Since the Russell class is a class it is significant that it belongs to itself simply because it is always significant for any class to be a member of any other class, classes being the kind of things which can significantly have members. However, individuals (or non-classes) are the kind of things which cannot significantly have members and hence $x\epsilon a$ is non-significant. $R\epsilon R$ is what I shall call <u>paradoxical</u>, since it takes the value $\frac{1}{2}$. So, the value $\frac{1}{2}$ is a significant value. This leads to a 4-valued logic with values 1, $\frac{1}{2}$, 0 and n, i.e. truth, paradoxicality, falsity and non-significance respectively. The main task of the thesis is to develop a 4-valued class theory incorporating both 3-valued theories, one with the value non-significance and one with the value paradoxicality, and to prove the consistency (relative to Z-F) of the whole theory.

CHAPTER I

SENTENTIAL LOGIC

In this chapter I will develop a 4-valued significance logic containing as sub-logics a 3-valued Lukasiewicz logic and a 3-valued significance logic.

1. The 3-Valued Significance Logic

The problem now is to decide the matrix representations of the connectives. For the connectives 'and', 'or', 'not', 'if ... then' and 'if and only if', it seems clear that criterion II(b) on page 239 of Goddard's paper ((10)) should apply. That is, "Any compound expression in which all the components are significant is itself significant." This just means that if no non-significance appears in a sentence then the laws of classical 2-valued logic apply to it. This criterion will henceforth be assumed unless otherwise stated.

The matrix for ~ is clear by the arguments presented in the Introduction. For example, both "The number 7 likes green cheese" and "The number 7 does not like green cheese" are non-significant. Hence the matrix for ~ is:-

~	
1	0
0	1
n	n

where '1' represents truth, '0' falsity and 'n' non-significance.

Now consider the connective 'and' by looking at the example "Saturday is in bed and this year is 1970". One would be inclined to say that it is non-significant on the grounds that one conjunct, "Saturday is in bed", is non-significant. Also, for the example "Saturday is in bed or this year is 1970" one could also say that it is non-significant on the grounds that one disjunct is non-significant. These three connectives would then satisfy Goddard's criterion II(a) (((10)) p. 239), "Any compound sentence with a non-significant component is non-significant." Applying this criterion to 'and' and 'or', we get the matrices:-

&	1	0	n
1	1	0	n
0	0	0	n
n	n	n	n

$\dot{\vee}$	1	0	n
1	1	1	n
0	1	0	n
n	n	n	n

One can similarly introduce matrices for 'if ... then' and 'if and only if' as follows:-

$\dot{\supset}$	1	0	n
1	1	0	n
0	1	1	n
n	n	n	n

$\dot{\equiv}$	1	0	n
1	1	0	n
0	0	1	n
n	n	n	n

The connectives $\sim$, $\&$, $\dot{\vee}$, $\dot{\supset}$ and $\dot{\equiv}$ all satisfy criteria II(a) and II(b) and are said by Goddard and Routley to be <u>classical connectives</u>. Once $\sim$ and $\&$ are introduced the rest can be defined in the same way as in classical 2-valued logic. That is, $p \mathbin{\dot{\vee}} q = \mathrm{df} \sim (\sim p \mathbin{\&} \sim q)$,

$p \mathbin{\dot{\supset}} q = \mathrm{df} \sim p \mathbin{\mathring{\vee}} q$ or $\sim(p \,\&\, \sim q)$, $p \mathbin{\mathring{\equiv}} q = \mathrm{df}\ (p \mathbin{\dot{\supset}} q) \,\&\, (q \mathbin{\dot{\supset}} p)$, where all sentential variables are unrestricted and 3-valued.

If the classical connectives are taken by themselves, then no logical law results if '1' is taken as the only designated value. This is because if 'n' is substituted for each sentential variable 'p', 'q', 'r', etc., in a formula, then the whole formula would take the value 'n'. If '1' and 'n' are both taken as designated values, as in Goddard (((10)) pp. 240 ff.), then unintuitive results follow, as pointed out in the paper. It seems a rash thing to designate non-significance anyway. So some connectives assuring us of some valid formulae are required to be added to the classical connectives.

Some operators which will do the job are the operators 'T' (it is true that), 'F' (it is false that) and 'S' (it is significant that), mentioned by Goddard (((10)) p. 237). He says,

> "To say of any sentence that it is true, that it is false or that it is non-significant, is to make a significant statement; and in particular, to say of a non-significant sentence that it is true, or that it is false, is to make a false statement."

Using this assumption, we get the following tables:-

T		F		S	
1	1	1	0	1	1
0	0	0	1	0	1
n	0	n	0	n	0

Goddard adopts this assumption, saying (p. 238), "We are using both

'T' and the variable to talk about sentences, and of all such sentences it is significant to say that they are true." Similarly, 'F' and 'S' are used to talk about sentences. So, I also wish to adopt the above tables for 'T', 'F' and 'S'.

Another connective which is needed when talking about significant and non-significant sentences is an equivalence with the meaning of 'has the same value as'. Goddard (pp. 240-1) uses the equivalence $\equiv$, which I will symbolise as '$\simeq$'.

$\simeq$	1	0	n
1	1	0	0
0	0	1	0
n	0	0	1

It is two-valued because saying that a sentence has the same value as another is saying something about the sentences rather than using them.

I would like to propose another disjunction other than the classical one which I think is useful formally and has some application in ordinary discourse. It is obtained from the criterion, "If one disjunct in a disjunction is non-significant then this disjunct is ignored when assessing the value of the disjunction." That is, if 'p' is non-significant, then '$p \vee q$' has the same value as 'q'. This gives the matrix:-

$\vee$	1	0	n
1	1	1	1
0	1	0	0
n	1	0	n

The disjunction is formally justified by regarding it as an existential quantification over a finite range. In the next chapter a quantifier S will be introduced with the property that the value of $(Sx)\phi(x)$ is assessed by examining only the significant $\phi(x)$'s and that $(Sx)\phi(x)$ is only non-significant if $\phi(x)$ is non-significant for all x. When assessing the value of $(Sx)\phi(x)$ from the significant $\phi(x)$'s the classical 2-valued logic applies, in accordance with Goddard's criterion II(b).

If one formalises "Something is happy" as $(Sx)Hx$, using the above quantifier S, then $(Sx)Hx$ will be true since some person in this world is happy and the quantifier has the effect of restricting the variable x to items of the "right" category (i.e. animals). "Something is happy" seems to me to be true because there is a happy person who is certainly a "thing" in whatever broad sense this has. However, if an existential quantifier E is used in place of S, where E satisfies Goddard's criterion II(a), so that $(Ex)\phi(x)$ is non-significant if $\phi(x)$ is non-significant for some x, then $(Ex)Hx$ is non-significant because it is non-significant for stones to be happy and "thing" would include all material objects. In fact, $(Ex)fx$ is always non-significant, except for rare predicates which have the universe as their significance range (e.g. "is a thing" ~~"exists"~~). Thus the quantifier S is necessary to represent "Something is happy" and to ensure the significance of existential statements. If the quantifier S is used over a finite range then $(Sx)\phi(x)$ would be equivalent to a finite disjunction where the connective would be 'V'.

Similarly to 'S' and 'V', one can introduce a universal quantifier ∀ and a conjunction + as follows:-

$$p + q = df \sim (\sim p \vee \sim q), \quad (\forall x)\phi(x) = df \sim (Sx) \sim \phi(x)$$

'+' is represented by the matrix:-

+	1	0	n
1	1	0	1
0	0	0	0
n	1	0	n

Similarly to $(Sx)\phi(x)$, the value of $(\forall x)\phi(x)$ is assessed by examining only the significant $\phi(x)$'s and $(\forall x)\phi(x)$ is only non-significant if $\phi(x)$ is non-significant for all x. Also, when assessing the value of $(\forall x)\phi(x)$ from the significant $\phi(x)$'s the classical 2-valued logic applies, in accordance with Goddard's criterion II(b).

Consider the example "Not all that glitters is gold". The intended meaning is that not all material objects that glitter are gold. By using the quantifier ∀ the sentence can be formalised without mention of material objects because of the automatic restriction to the significance range of "x glitters" and "x is gold". If the quantifier A is used in place of ∀, where A satisfies Goddard's criterion II(a), so that $(Ax)\phi(x)$ is non-significant if $\phi(x)$ is non-significant for some x, then the variable x must be restricted to material objects otherwise the sentence formalised as "$\sim (Ax)(Gl\,x \supset Gd\,x)$" will be non-significant which it clearly is not. So the quantifier $\forall$ allows a more direct and natural formalisation of "Not all that glitters is gold". Again, if such quantification ranges over a finite domain then it can be replaced by a finite conjunction using the connective, +.

There is another use of the disjunction, $\vee$. Consider the compound predicate, "is a holiday or likes cheese". If "x is a holiday" is true (say, x is New Year's Day) then "x is a holiday or likes cheese" is true. Similarly, if "x likes cheese" is true (say, x is Jerry, the mouse) then "x is a holiday or likes cheese" is true. If "x is a holiday" and "x likes cheese" are both non-significant (say, x is a piece of wood) then "x is a holiday or likes cheese" is non-significant. If "x is a holiday" is false and "x likes cheese" is not true (say, x is my birthday) then "x is a holiday or likes cheese" is false. Letting "x is a holiday" be 'p', "x likes cheese" be 'q' and "x is a holiday or likes cheese" be 'pDq', the matrix for D as determined above is:-

D	1	0	n
1	1	1	1
0	1	0	0
n	1	0	n

which is exactly the matrix for $\vee$. Thus 'fx $\vee$ gx' has the same value as '(f or g)x' and the disjunction $\vee$ can be used in representing a predicate disjunction. This does not mean that '(f or g)x' can be interpreted as 'fx or gx', where the 'or' is a classical sentential connective. The sentence "x is a holiday or x likes cheese" is always non-significant because whenever "x is a holiday" is significant, "x likes cheese" is non-significant, and vice versa.

Thus I have shown that the disjunction V is formally useful and has some application in ordinary discourse. However it can be defined in terms of other connectives which I will show are essential to the formal theory.

Although, as already pointed out, restricted variables are not always necessary because of the type of quantification where variables are automatically restricted to the significance ranges of their predicates, there are many cases where they are necessary. Within the framework of a general system with variables ranging over everything or over a wide range, one may want to restrict the theory to a particular context. For example, in a theory of sets or classes, one may want to restrict consideration to ordinals, cardinals, integers, etc. To do this formally, one has to restrict the quantifiers to the required class of things so that all the original logical laws of the general system still hold in the restricted system.

In the 2-valued predicate calculus restrictions are placed on the variables as follows:-

$$(Ax)\,\phi(x) = \mathrm{df}\ (AX)(B(X) \supset \phi(X)),$$

$$(Sx)\,\phi(x) = \mathrm{df}\ (SX)\ (B(X)\ \&\ \phi(X)),$$

provided (SX) B(X) is valid.

'x' is the restricted variable and 'X' is the unrestricted variable. 'x' is restricted to those X's such that B(X) is true, given that there is at least one such X. In the 2-valued case, '$\supset$' and '&' are the essential connectives for this restriction of variable.

'$\supset$' satisfies the condition that if B(X) is true then (B(X) $\supset \phi$(X) is equivalent to ϕ(X) and if B(X) is false then B(X) $\supset \phi$(X) is true. For a particular X_0, if $B(X_0) \supset \phi(X_0)$ is true then the value of (AX) (B(X) $\supset \phi$(X)), which is determined from all the values of the B(X) $\supset \phi$(X)'s, is the same whether $B(X_0) \supset \phi(X_0)$ is considered in the valuation or not. So, whenever B(X) is false, B(X) $\supset \phi$(X) is ignored when assessing the value of (AX) (B(X) $\supset \phi$(X)). Since there is at least one X such that B(X) is true, not all of the B(X) $\supset \phi$(X)'s are ignored. Hence, to evaluate (AX) (B(X) $\supset \phi$(X)), one only has to consider the X's such that B(X) is true and the values of ϕ(X) for these X's.

'&' satisfies the condition that if B(X) is true then B(X) & ϕ(X) is equivalent to ϕ(X) and if B(X) is false then B(X) & ϕ(X) is false. For a particular X_0, if $B(X_0)$ & $\phi(X_0)$ is false then the value of (SX) (B(X) & ϕ(X)), which is determined from all the values of the B(X) & ϕ(X)'s, is the same whether $B(X_0)$ & $\phi(X_0)$ is considered in the valuation or not. So, whenever B(X) is false, B(X) & ϕ(X) is ignored when assessing the value of (SX) (B(X) & ϕ(X)). Since there is at least one X such that B(X) is true, not all of the B(X) & ϕ(X)'s are ignored. Hence, to evaluate $(SX)\ (B(X)\ \&\ \phi(X))$, one only has to consider the X's such that $B(X)$ is true and the values of $\phi(X)$ for these X's.

In the 3-valued predicate logic we must find similar connectives to the '$\supset$' and '&' of the 2-valued logic to restrict the variables. As previously remarked, the quantifiers of the 3-valued logic are essentially A and S, which are defined as follows:-

$(AX)\phi(X)$ is true iff $\phi(X)$ is true for all X; $(AX)\phi(X)$ is non-significant iff $\phi(X)$ is non-significant for some X.

$(SX)\ \phi\ (X)$ is true iff $\phi(X)$ is true for some X; $(SX)\phi(X)$ is non-significant iff $\phi(X)$ is non-significant for all X.

Note that E and $\overline{V}$ can be defined as follows:-

$$(EX)\phi(X) = df \sim (AX) \sim \phi(X),$$

$$(\overline{V}X)\phi(X) = df \sim (SX) \sim \phi(X).$$

The 3-valued connectives required are $\supset$ and $\bar{\&}$, defined as follows:-

$\supset$	1	0	n
1	1	0	n
0	1	1	1
n	1	1	1

$\bar{\&}$	1	0	n
1	1	0	n
0	n	n	n
n	n	n	n

'$\supset$' satisfies the property that if $B(X)$ is true, $B(X) \supset \phi(X)$ is equivalent to $\phi(X)$, and if $B(X)$ is not true, $B(X) \supset \phi(X)$ is true. For a particular X_o, if $B(X_o) \supset \phi(X_o)$ is true then the value of (AX) $(B(X) \supset \phi(X))$, which is determined from all the values of the $B(X)$ $\supset \phi(X)$'s, is the same whether $B(X_o) \supset \phi(X_o)$ is considered in the valuation or not. So, whenever B(X) is not true, B(X) $\supset \phi$(X) is ignored when assessing the value of (AX) (B(X) $\supset \phi$(X)). Since there is at least one X such that (B(X) is true, not all of the B(X) $\supset \phi$(X)'s are ignored. Hence, to evaluate (AX) (B(X) $\supset \phi$(X)), one only has to consider the X's such that B(X) is true and the values of ϕ(X) for these X's.

'$\bar{\&}$' satisfies the property that if B(X) is true, B(X) $\bar{\&}$ ϕ(X) is equivalent to ϕ(X), and if B(X) is not true, B(X) $\bar{\&}$ ϕ(X) is non-significant. For a particular X_o, if B(X_o) $\bar{\&}$ ϕ(X_o) is non-significant, then the value of (SX) (B(X) & ϕ(X)), which is determined from all the values of the B(X) & ϕ(X)'s, is the same whether B(X_o) $\bar{\&}$ ϕ(X_o) is considered in the valuation or not. So, whenever B(X) is not true, B(X) $\bar{\&}$ ϕ (X) is ignored when assessing the value of (SX) (B(X) $\bar{\&}$ ϕ (X)). Since there is at least one X such that B(X) is true, not all of the B(X) $\bar{\&}$ ϕ (X)'s are ignored. Hence, to evaluate (SX) (B(X) $\bar{\&}$ ϕ (X)), one only has to consider the X's such that B(X) is true and the values of ϕ(X) for these X's.

Since these properties uniquely determine the connectives $\supset$ and $\bar{\&}$, these are the only connectives that can satisfactorily restrict variables when the quantifiers A and S are used.

I will show later on that if all variables are restricted using a given predicate, B, say, such that (SX)B(X) is valid, then all the axioms and rules of the 3-valued predicate logic will be preserved.

I will introduce a monadic operator, T_n, such that its matrix is as follows:-

T_n	
1	1
0	n
n	n

$\bar{\&}$ and T_n are interdefinable using $\supset$ and &. $T_n p = \text{df}\ p \,\bar{\&}\, (p \supset p)$; $p \,\bar{\&}\, q = \text{df}\ T_n p \,\&\, q$. It is to be noted that $\bar{\&}$ and T_n are the only connectives introduced that fail to satisfy Goddard's criterion II(b).

I will now show that the set of primitives $\{\sim, \supset, T_n\}$ yields a functionally complete system. Firstly, I will show that these primitives are independent.

(i) Let $A(p)$ be defined in terms of $\supset$ and T_n only. If p has the value 1, then $A(p)$ has the value 1. Hence $\sim$ is not definable in terms of $\supset$ and T_n only.

(ii) $\supset$ cannot be defined in terms of monadic operators only.

(iii) Let $A(p)$ be defined in terms of $\sim$ and $\supset$ only. If p has the value 0, then $A(p)$ has the value 1 or 0. Hence T_n is not definable in terms of $\sim$ and $\supset$ only.

This has shown that the primitives $\sim$, $\supset$ and T_n are independent. Now I wish to introduce the following definitions:-

$$tp = \text{df}\ p \supset p$$

$$fp = \text{df}\ \sim t\,p$$

$$np = \text{df}\ T_n \sim tp$$

$$Tp = df \sim (p \supset fp)$$

$$Fp = df\ T \sim p$$

$$p \ddot{v} q = df\ (p \supset q) \supset q$$

$$p \ddot{\&} q = df \sim (\sim p \ddot{v} \sim q)$$

$$Sp = df\ T p \ddot{v} F p$$

$$p \& q = df\ (Sp \ddot{\&} Sq \supset . p \ddot{\&} q) \ddot{\&} (\sim Sp \ddot{v} \sim Sq \supset np)$$

The matrices for these are as follows:-

t	
1	1
0	1
n	1

f	
1	0
0	0
n	0

n	
1	n
0	n
n	n

T	
1	1
0	0
n	0

F	
1	0
0	1
n	0

$\ddot{v}$	1	0	n
1	1	1	1
0	1	0	n
n	1	0	n

$\ddot{\&}$	1	0	n
1	1	0	n
0	0	0	0
n	1	0	n

S	
1	1
0	1
n	0

&	1	0	n
1	1	0	n
0	0	0	n
n	n	n	n

Consider the n-adic connective, $\Delta(p_1, \text{--}, p_n)$. For each assignment of values to $p_1, \text{--}, p_n$, $\Delta(p_1, \text{--}, p_n)$ will take some value, 1, 0 or n. Let $i_1, \text{--}, i_n$ be the values of a particular assignment for $p_1, \text{--}, p_n$, respectively. Let $\lesseqgtr_{i_1, \text{--}, i_n}$ be tp, fp or np according to the value (1, 0 or n, respectively) of $\Delta(p_1, \text{--}, p_n)$ under the assignment $i_1, \text{--}, i_n$. For each assignment of values to $p_1, \text{--}, p_n$ there will be a formula, $K_1 p_1 \& K_2 p_2 \& \text{----} \& K_n p_n \supset \lesseqgtr_{i_1, \text{--}, i_n}$, where K_j is T, F or ~S according to the value, 1, 0 or n, respectively, of i_j. $\Delta(p_1, \text{--}, p_n)$ can now be defined as the conjunction, using &, of all of these formulae, one formula corresponding to one assignment of values to $p_1, \text{--}, p_n$. Given a particular assignment, one and only one expression of the form $K_1p_1 \,\&\, K_2p_2 \,\&\, \text{----} \,\&\, K_np_n$ will be true while all other such expressions will be false. The expression $K_1p_1 \,\&\, \text{----} \,\&\, K_np_n$ which is true will be part of the formula $K_1p_1 \,\&\, K_2p_2 \,\&\, \text{----} \,\&\, K_np_n \supset \leqslant_{i_1, \text{--}, i_n}$ which corresponds to the particular assignment. All other conjuncts in the conjunction defined to be $\Lambda(p_1, \text{--}, p_n)$ will be true, while this conjunct will take the value of $\leqslant_{i_1, \text{--}, i_n}$, which is the value of $\Lambda(p_1, \text{--}, p_n)$ under the given assignment. Hence the above definition of $\Lambda(p_1, \text{--}, p_n)$ is satisfactory. Hence all connectives can be defined and the primitives $\sim$, $\supset$ and T_n form a functionally complete system.

Now I will give a formal axiomatic system for the 3-valued significance logic with the above primitives. But first, I will give an axiomatisation of the 2-valued logic, which I will call system P.

System P

Primitives

1. $p, q, r,$ ------ (2-valued sentential variables)
2. $\sim, \supset$. (negation and implication connectives)

Formation Rules

1. A 2-valued sentential variable is a wff.
2. If A and B are wffs, then $\sim A$ and $(A \supset B)$ are wffs.

Definitions

1. $A \& B =_{df} \sim (A \supset \sim B)$

2. $A \vee B =_{df} \sim A \supset B$

3. $A \equiv B =_{df} (A \supset B) \& (B \supset A)$

4. $TA =_{df} A$

5. $FA =_{df} T \sim A$

6. $CA =_{df} TA \vee FA$

7. $SA =_{df} TA \vee \sim TA$

8. $PA =_{df} SA \& \sim CA$

Axioms

1. $p \supset (q \supset p)$

2. $p \supset (q \supset r) \supset . p \supset q \supset . p \supset r$

3. $\sim p \supset \sim q \supset . q \supset p$

Rules

1. Substitution for sentential variables.

2. $\vdash_P A, \vdash_P A \supset B \Rightarrow \vdash_P B$

The 2-valued system P is complete.

System S

The system P above is used in the construction of system S, but it is not a subsystem of system S. Note that the two systems have the common symbols p, q, r, etc., for sentential variables, and $\sim$ and

$\supset$ for two connectives. It will be clear from the context which system the symbol is being used in and if there is any doubt, the appropriate wff will be prefixed by an S or P, accordingly. This will be done by using an assertion sign, e.g. $\vdash_P$, $\vdash_S$, or by using the two dots, e.g. P: $p \supset p \vee q$.

The 3-valued formal axiomatic system S is as follows:-

Primitives

1. p, q, r, ------ (3-valued sentential variables)
2. $\sim$, $\supset$, T_n (connectives)

Formation Rules

1. A sentential variable is a wff.
2. If A and B are wffs, then $\sim A$, $A \supset B$ and $T_n A$ are wffs.

Since $\{\sim, \supset, T_n\}$ is functionally complete all connectives are definable. The ones I will be using are given by the following matrices:-

$\sim$	
1	0
0	1
n	n

$\supset$	1	0	n
1	1	0	n
0	1	1	1
n	1	1	1

T_n	
1	1
0	n
n	n

T	
1	1
0	0
n	0

F	
1	0
0	1
n	0

S	
1	1
0	1
n	0

&	1	0	n
1	1	0	n
0	0	0	n
n	n	n	n

$\dot{v}$	1	0	n
1	1	1	n
0	1	0	n
n	n	n	n

$\dot{\supset}$	1	0	n
1	1	0	n
0	1	1	n
n	n	n	n

$\mathring{\equiv}$	1	0	n
1	1	0	n
0	0	1	n
n	n	n	n

$\mathring{\vee}$	1	0	n
1	1	1	1
0	1	0	0
n	1	0	n

$+$	1	0	n
1	1	0	1
0	0	0	0
n	1	0	n

$\simeq$	1	0	n
1	1	0	0
0	0	1	0
n	0	0	1

$\equiv$	1	0	n
1	1	0	n
0	0	1	1
n	n	1	1 *

They can be defined in terms of the primitives as follows:-

$Tp = df \sim (p \supset \sim (p \supset p))$

$Fp = df\ T \sim p$

$p \ddot{\vee} q = df\ (p \supset q) \supset q$

$p \ddot{\&} q = df \sim (\sim p \ddot{\vee} \sim q)$

$\bar{S}p = df\ Tp \ddot{\vee} Fp$

$p \& q = df\ (\bar{S}p \ddot{\&} \bar{S}q \supset . p \ddot{\&} q) \ddot{\&} (\sim \bar{S}p \ddot{\vee} \sim \bar{S}q \supset T_n \sim (p \supset p))$

$p \vee q = df\ (p \ddot{\vee} q) \ddot{\&} (q \ddot{\vee} p)$

$Sp = df\ Tp \vee Fp$

$p \mathring{\vee} q = df \sim (\sim p \& \sim q)$

$p \dot{\supset} q = df \sim (p \& \sim q)$

$p \mathring{\equiv} q = df\ (p \dot{\supset} q) \& (q \dot{\supset} p)$

$p + q = df \sim (\sim p \vee \sim q)$

---

* In these matrices, as in all the matrices introduced in this thesis, the only designated value is '1'.

$p \equiv q = \text{df } (p \supset q) \& (q \supset p)$

$p \approx q = \text{df } (Tp \& Tq) \vee (Fp \& Fq) \vee (\sim Sp \& \sim Sq)$

Axioms

1. STp
2. $\sim Sp \supset \sim S \sim p$
3. $Tp \supset TT_n p$
4. $Fp \supset \sim ST_n p$
5. $\sim Sp \supset \sim ST_n p$
6. $\sim Sp \supset T(p \supset q)$
7. $Fp \& \sim Sq \supset T(p \supset q)$
8. $Tp \& \sim Sq \supset \sim S(p \supset q)$

Rules

1. Substitution for sentential variables.
2. $\vdash_S A,\ \vdash_S A \supset B \Rightarrow \vdash_S B$
3.* $\vdash_P A(p_1, \text{---}, p_n) \Rightarrow \vdash_S SB_1 \supset . \text{---} \supset . SB_n \supset A(B_1, \text{---}, B_n)$, where $p_1, \text{---}, p_n$ are all the variables in A.

It is in this Rule 3 that one needs to assume the theses of system P. It is not a rule in the usual sense, where one could obtain theses of S from other theses of S. It is more of a generalised axiom. In fact, the reason for its being in the form it is, i.e. as opposed to:

$$\vdash_P A(p_1, \text{---}, p_n),\ \vdash_S SB_1, \text{---}, \vdash_S SB_n \Rightarrow \vdash_S A(B_1, \text{---}, B_n),$$

* This rule is the idea of Prof. R. Routley.

where $p_1, \text{--}, p_n$ are all of the variables in A, is so that the Deduction Theorem is made easier to prove, in that $S\,B_1 \supset . \text{---} \supset A\,(B_1, \text{--}, B_n)$ can be treated as an axiom scheme of S.

The function of the rule is to preserve the 2-valued matrices for the connectives, so that the 3-valued matrices are extensions of the 2-valued matrices.

Theorems

1. $\underline{S\,F\,p}$

A.1., R.1., Defn. F : $S\,F\,p$.

('A' is for axioms, 'R' is for rules, 'T' is for theorems of S, and 'P' for theorems of P.)

2. $\underline{S\,S\,p}$

P : $S(p \vee q)$ --- (1)

A.1., T.1., (1), R.3 : $S(T\,p \vee F\,p)$ --- (2).

(2), Defn. S : $S\,S\,p$.

3. $\underline{F\,p \supset T \sim p}$

P : $p \supset p$ --- (1)

(1), T.1., R.3 : $T \sim p \supset T \sim p$ --- (2)

(2), Defn. F : $F\,p \supset T \sim p$.

4. $\underline{T\,p \supset S\,p}$

P : $p \supset p \vee q$ --- (1)

A.1, T.1., (1), R.3 : $T\,p \supset T\,p \vee F\,p$ --- (2)

(2), Defn. S : $T\,p \supset S\,p$.

5. <u>$T\,p \supset F \sim p$</u>

P : $T\,p \supset F \sim p$ --- (1)

(1), R.3 : $S\,p \supset . T\,p \supset F \sim p$ --- (2)

(2), P, R.3 : $S\,p \,\&\, T\,p \supset F \sim p$ --- (3)

T.4, P, R.3 : $T\,p \supset S\,p \,\&\, T\,p$ --- (4)

(3),(4), P, R.3 : $T\,p \supset F \sim p$.

6. <u>$F\,p \supset S\,p$</u>

A.2., P, R.3 : $S \sim p \supset S\,p$ --- (1)

T.4, R.1, Defn. F : $F\,p \supset S \sim p$ --- (2)

(1),(2), P, R.3 : $F\,p \supset S\,p$.

7. <u>$T\,p \,\&\, T\,q \supset T\,(p \supset q)$</u>

P : $T\,p \,\&\, T\,q \supset T\,(p \supset q)$ --- (1)

(1), R.3 : $S\,p \supset . S\,q \supset . T\,p \,\&\, T\,q \supset T\,(p \supset q)$ --- (2)

(2), P, R.3 : $(T\,p \,\&\, S\,p) \,\&\, (T\,q \,\&\, S\,q) \supset T(p \supset q)$ --- (3)

T.4, P, R.3 : $T\,p \supset T\,p \,\&\, S\,p$ --- (4)

T.4, P, R3 : $T\,q \supset T\,q \,\&\, S\,q$ --- (5)

(4),(5), P, R.3 : $T\,p \,\&\, T\,q \supset (T\,p \,\&\, S\,p) \,\&\, (T\,q \,\&\, S\,q)$ --- (6)

(3),(6), P, R.3 : $T\,p \,\&\, T\,q \supset T\,(p \supset q)$

8. <u>$T\,p \,\&\, F\,q \supset F\,(p \supset q)$</u>

P : $T\,p \,\&\, F\,q \supset F\,(p \supset q)$ --- (1)

(1), R.3 : $S\,p \supset . S\,q \supset . T\,p \,\&\, F\,q \supset F\,(p \supset q)$ --- (2)

(2), P, R.3 : $(T\,p \,\&\, S\,p) \,\&\, (F\,q \,\&\, S\,q) \supset F\,(p \supset q)$ --- (3)

T.4, P, R.3 : $T\,p \supset T\,p \,\&\, S\,p$ --- (4)

T.6, P, R.3 : $F\,q \supset F\,q \,\&\, S\,q$ --- (5)

(4),(5), P, R.3: $Tp \,\&\, Fq \supset (Tp \,\&\, Sp) \,\&\, (Fq \,\&\, Sq)$ --- (6)

(3),(6), P, R.3: $Tp \,\&\, Fq \supset F(p \supset q)$

9. $\underline{Fp \,\&\, Sq \supset T(p \supset q)}$

P : $Fp \,\&\, Sq \supset T(p \supset q)$ --- (1)

(1), R.3 : $Sp \supset . Sq \supset . Fp \,\&\, Sq \supset T(p \supset q)$ --- (2)

(2), P, R.3 : $(Fp \,\&\, Sp) \,\&\, Sq \supset T(p \supset q)$ --- (3)

T.6, P, R.3 : $Fp \supset Fp \,\&\, Sp$ --- (4)

(3),(4), P, R.3 : $Fp \,\&\, Sq \supset T(p \supset q)$

10. $\underline{Tp \vee Fp \vee \sim Sp}$

P : $p \vee \sim p$ --- (1)

(1), R.3 : $(Tp \vee Fp) \vee \sim (Tp \vee Fp)$ --- (2)

(2), Defn. S : $Tp \vee Fp \vee \sim Sp$

11. $\underline{(Tp \supset Tq) \,\&\, (Fp \supset Tq) \,\&\, (\sim Sp \supset Tq) \supset Tq}$

P : $(p \supset s) \,\&\, (q \supset s) \,\&\, (r \supset s) \supset (p \vee q \vee r \supset S)$ --- (1)

(1), R.3 : $(Tp \supset Tq) \,\&\, (Fp \supset Tq) \,\&\, (\sim Sp \supset Tq)$

$\supset (Tp \vee Fp \vee \sim Sp \supset Tq)$ --- (2)

(2), T.10, P, R.3 : $(Tp \supset Tq) \,\&\, (Fp \supset Tq)$

$\,\&\, (\sim Sp \supset Tq) \supset Tq$

D.R.1. $\underline{\vdash_S TA \Rightarrow \vdash_S A}$

T.4, R.2 : S A --- (1)

P : $Tp \supset p$ --- (2)

(1),(2), R.3 : $TA \supset A$ --- (3)

(3) : A

('D.R.' is used for derived rules.)

Although the above theses and derived rules are sufficient to prove completeness, I want to add an example of a 3-valued theorem without any 2-valued operators, T, F or S, in it. The reason for this is that there may be some doubt as to how such a theorem can be derived from axioms in which every connective which involves the 3 values is covered by a 2-valued operator. The proof will proceed in a similar fashiom to that used in the completeness proof.

12. $\underline{(p \supset . q \supset \sim q) \supset (q \supset . p \supset \sim p)}$

T.5 : $T\,p \supset F \sim p$ —— (1)

T.8 : $T\,p \,\&\, F \sim p \supset F\,(p \supset \sim p)$ —— (2)

P : $p \supset q \supset . p \supset p \,\&\, q$ —— (3)

(3), R.3 : $T\,p \supset F \sim p \supset . T\,p \supset T\,p \,\&\, F \sim p$ —— (4)

(1),(2),(4), P, R.3 : $T\,p \supset F\,(p \supset \sim p)$ —— (5)

A.7, T.9, P, R.3 : $F\,p \supset T\,(p \supset q)$ —— (6)

(6) : $F\,p \supset T\,(p \supset \sim p)$ —— (7)

A.6 : $\sim S\,p \supset T\,(p \supset \sim p)$ —— (8)

(6) : $F\,q \supset T\,(q \supset . p \supset \sim p)$ —— (9)

A.6 : $\sim S\,q \supset T\,(q \supset . p \supset \sim p)$ —— (10)

(5), T.3, P, R.3 : $T\,p \,\&\, T\,q \supset F\,(q \supset . p \supset \sim p)$ —— (11)

(7), T.7, P, R.3 : $F\,p \,\&\, T\,q \supset T\,(q \supset . p \supset \sim p)$ —— (12)

(10), T.7, P, R.3 : $\sim S\,p \,\&\, T\,q \supset T\,(q \supset . p \supset \sim p)$ —— (13)

(11), T.6, T.9, P, R.3 : $T\,p \,\&\, T\,q \supset T$ (T.12) —— (14)

(9),(12), T.7, P, R.3 : $T\,p \,\&\, F\,q \supset T$ (T.12) —— (15)

(10),(13), T.7, P, R.3 : $T\,p \,\&\, {\sim} S\,q \supset T$ (T.12) —— (16)

(9),(12), T.7, P, R.3 : $F\,p \,\&\, T\,q \supset T$ (T.12) —— (17)

(9), T.7, P, R.3 : $F\,p \,\&\, F\,q \supset T$ (T.12) —— (18)

(9),(10), T.7, P, R.3 : $F\,p \,\&\, {\sim} S\,q \supset T$ (T.12) —— (19)

(10),(13), T.7, P, R.3 : ${\sim} S\,p \,\&\, T\,q \supset T$ (T.12) —— (20)

(9),(10), T.7, P, R.3 : ${\sim} S\,p \,\&\, F\,q \supset T$ (T.12) —— (21)

(10), T.7, P, R.3 : ${\sim} S\,p \,\&\, {\sim} S\,q \supset T$ (T.12) —— (22)

T.10, P, R.3 : T (T.12) —— (23)

(23), D.R.1. : T.12.

Now I give a completeness proof for the set of axioms and rules with respect to the 3-valued matrices. The proof is adapted from one given by Church ((2)) pp. 97-99) for the 2-valued sentential logic.* First I will prove a lemma.

Lemma

Let B be a wff of S and let $p_1, \text{---}, p_n$ be the distinct variables occurring in B. Let A_i be Tp_i, Fp_i or ${\sim}Sp_i$ according as the value a_i of p_i is 1, 0 or n. Let B' be TB, FB or ~SB according as the value of B, for values a_1, a_2, ---, a_n of p_1, p_2, ---, p_n is 1, 0, or n. Then $\vdash_S A_1 \,\&\, \text{--} \,\&\, A_n \supset B'$.

Proof

By induction on the number of variable occurrences.

It is trivial in the case of a single variable.

---

*This adaptation was suggested by Prof. R. Routley.

(i) <u>B is $\sim B_1$</u>

By ind. hyp., A_1 & — & $A_n \supset B_1'$. If B_1 has the value 1, 0 or n, then B has the value 0, 1 or n, respectively. So, if B_1' is TB_1, FB_1 or $\sim SB_1$, then B' is $F \sim B_1$, $T \sim B_1$ or $\sim S \sim B_1$, respectively.

By T.5, T.3 and A.2, $TB_1 \supset F \sim B_1$, $FB_1 \supset T \sim B_1$, and $\sim SB_1 \supset \sim S \sim B_1$. Hence $B_1' \supset B'$ and, by using P and R.3, A_1 & — & $A_n \supset B'$.

(ii) <u>B is $B_1 \supset B_2$</u>

By ind. hyp., A_1 & — & $A_n \supset B_1'$ and A_1 & — & $A_n \supset B_2'$, where $p_1, \text{--}, p_n$ are all the distinct variables in $B_1 \supset B_2$.

(a) If B_1 has the value 0 or n, then B has the value 1. So, if B_1' is FB_1 or $\sim SB_1$, then B' is $T\ (B_1 \supset B_2)$.
By T.9, $F\ p$ & $S\ q \supset T\ (p \supset q)$ and by A.7, $F\ p$ & $\sim S\ q \supset T\ (p \supset q)$. By using P and R.3, $F\ p \supset T\ (p \supset q)$. By A.6, $\sim S\ p \supset T\ (p \supset q)$. Hence $B_1' \supset B'$ and by using P and R.3, A_1 & — & $A_n \supset B'$.

(b) If B_1 has the value 1, then if B_2 has the value 1, 0 or n then B has the value 1, 0 or n, respectively. So, if B_1' is TB_1, then if B_2' is TB_2, FB_2 or $\sim SB_2$ then B' is $T\ (B_1 \supset B_2)$, $F\ (B_1 \supset B_2)$ or $\sim S\ (B_1 \supset B_2)$, respectively.
By T.7, TB_1 & $TB_2 \supset T\ (B_1 \supset B_2)$.
By T.8, TB_1 & $FB_2 \supset F\ (B_1 \supset B_2)$.
By A.8, TB_1 & $\sim SB_2 \supset \sim S\ (B_1 \supset B_2)$.
Hence B_1' & $B_2' \supset B'$. Since $p \supset (q \supset p\ \&\ q)$ holds in P,

by using R.3, A_1 & — & $A_n \supset B_1'$ & B_2' and hence A_1 & — & $A_n \supset B'$.

(iii) B is $T_n B_1$

By ind. hyp., A_1 & — & $A_n \supset B_1'$.

If B_1 has the value 1, 0 or n, then B has the value 1, n or n, respectively. So, if B_1' is TB_1, FB_1 or $\sim SB_1$, then B' is $T T_n B_1$, $\sim S T_n B_1$ or $\sim S T_n B_1$, respectively.

By A.3, A.4 and A.5, $TB_1 \supset T T_n B_1$, $FB_1 \supset_1 \sim S T_n B_1$ and $\sim SB_1 \supset \sim S T_n B_1$. Hence $B_1' \supset B'$ and, by using P and R.3, A_1 & — & $A_n \supset B'$.

Meta-theorem 1

If B is valid according to the 3-valued matrices, then B is a thesis of the axiomatic system S.

Proof

Let $p_1, —, p_n$ be the distinct variables of B. Let $A_1, —, A_n$ be as in the lemma. Since B is valid, B' is TB, independently of the values of $p_1, —, p_n$. Hence, $\vdash_S A_1$ & — & A_{n-1} & $T p_n \supset T B$. By P and R.3, $\vdash_S A_1$ & — & $A_{n-1} \supset (T p_n \supset T B)$. Similarly, $\vdash_S A_1$ & — & $A_{n-1} \supset (F p_n \supset T B)$ and $\vdash_S A_1$ & — & $A_{n-1} \supset (\sim S p_n \supset T B)$. By P and R.3, $\vdash_S A_1$ & — & $A_{n-1} \supset . (T p_n \supset T B)$ & $(F p_n \supset T B)$ & $(\sim S p_n \supset T B)$. Using T.11, $\vdash_S A_1$ & — & $A_{n-1} \supset T B$. By repeating this procedure for each A_i, $1 \leqslant i \leqslant n-1$, $\vdash_S T B$. Hence, by D.R.1, $\vdash_S B$.

Meta-theorem 2

The Deduction Theorem holds in S for $\supset$. That is, if $A_1, \text{---}, A_n \vdash_S B$ then $A_1, \text{---}, A_{n-1} \vdash_S A_n \supset B$, provided no substitution is made on variables of A_n.

Proof

Let there be a proof of B from $A_1, \text{---}, A_n$ using the axioms and rules of S. Place '$A_n \supset$' in front of every step in this proof. If A_n is used in the proof of B, then $\vdash_S A_n \supset A_n$. Since $\vdash_S p \supset (q \supset p)$, by Meta-theorem 1, $\vdash_S A_n \supset$ any axiom. By R.1, if $\vdash_S C\ (p) \Rightarrow \vdash_S C\ (D)$ then $\vdash_S A_n \supset C\ (p) \Rightarrow \vdash_S A_n \supset C\ (D)$. Since $\vdash_S (p \supset q) \supset . (p \supset . q \supset r) \supset . p \supset r$, by Meta-theorem 1, if $\vdash_S C$, $\vdash_S C \supset D \Rightarrow \vdash_S D$ then $\vdash_S A_n \supset C$, $\vdash_S A_n \supset (C \supset D) \Rightarrow \vdash_S A_n \supset D$. If $\vdash_P A\ (p_1, \text{---}, p_n) \Rightarrow \vdash_S S\,B_1 \supset . \text{---} \supset . S\,B_n \supset A\ (B_1, \text{---}, B_n)$, then, since $\vdash_S p \supset (q \supset p)$, $\vdash_S A_n \supset . S\,B_1 \supset . \text{---} \supset . S\,B_n \supset A\ (B_1, \text{---}, B_n)$. Hence, '$A_n \supset$' can be inserted in front of every step in the proof of B from $A_1, \text{---}, A_n$ and the theorem follows.

Meta-theorem 3

Substitutivity of Equivalents holds in S for $\approx$. That is, if $\vdash_S A \approx B$ then $\vdash_S C\ (A) \approx C\ (B)$, where substitution into C can be made for any particular argument place.

Proof

By Meta-theorem 1, $\vdash_S A \approx B \supset . \sim A \approx \sim B$, $\vdash_S A \approx B \supset . D \supset A \approx$.

$D \supset B$, $\vdash_S A \simeq B \supset . A \supset D \simeq . B \supset D$, $\vdash_S A \simeq B \supset . T_n A \simeq T_n B$. By applying these to each connective of C in turn, $\vdash_S C(A) \simeq C(B)$ can be shown.

Meta-theorem 4

Let B be a wff of S containing only the connectives, $\supset$, T_n, T, &, v, $\sim T$. (If $\sim$ occurs at all, it must precede a T.) Let A be a wff of P obtained from B by deleting any T's or T_n's and replacing $\sim T$ by $\sim$. Then, if $\vdash_P A$ then $\vdash_S B$.

Proof

The proof is similar to that of Meta-theorem 1, except that there are only two values, 1 and not-1. The lemma can be stated as follows:- Let B be a wff of S and let $p_1, \text{--}, p_n$ be the distinct variables occurring in B. Let A_i by Tp_i or $\sim Tp_i$ according as the value a_i of p_i is 1 or not-1. Let B' be TB or $\sim$TB according as the value of B, for values $a_1, \text{--}, a_n$ of $p_1, \text{--}, p_n$, is 1 or not-1. Then $\vdash_S A_1 \,\&\, \text{--} \,\&\, A_n \supset B'$. This is proved similarly to the lemma, making use of Meta-theorem 1. The proof of Meta-theorem 4 is similar to that of Meta-theorem 1. One can prove $\vdash_S A_1 \,\&\, \text{--} \,\&\, A_{n-1} \supset . Tp_n \supset T\,B$ and $\vdash_S A_1 \,\&\, \text{--} \,\&\, A_{n-1} \supset . \sim Tp_n \supset T\,B$. By using Meta-theorem 1, $\vdash_S A_1 \,\&\, \text{--} \,\&\, A_{n-1} \supset T\,B$ and by repetition $\vdash_S T\,B$. Hence $\vdash_S B$.

Meta-theorem 5

Let C(p) be a wff of S containing only the connectives, $\supset$, T_n, T, &, $\vee$, $\sim T$, as in Meta-theorem 4. Then, if $\vdash_S A \equiv B$ then $\vdash_S C(A) \equiv C(B)$, with substitution as in Theorem 3. (A and B can contain any connectives.)

Proof

By Meta-theorem 1, $\vdash_S A \equiv B \supset . D \supset A \equiv . D \supset B$, $\vdash_S A \equiv B \supset . A \supset D \equiv . B \supset D$, $\vdash_S A \equiv B \supset . T_n A \equiv T_n B$, $\vdash_S A \equiv B \supset . T A \equiv T B$, $\vdash_S A \equiv B \supset . A \& D \equiv B \& D$, $\vdash_S A \equiv B \supset . D \& A \equiv D \& B$, $\vdash_S A \equiv B \supset . A \vee D \equiv B \vee D$, $\vdash_S A \equiv B \supset . D \vee A \equiv D \vee B$, $\vdash_S A \equiv B \supset . \sim T A \equiv \sim T B$. By applying these to each connective of C in turn, $\vdash_S C(A) \equiv C(B)$ can be shown.

. . .

This completes the account of this 3-valued significance logic. Another axiomatisation appears in Rosser and Turquette's Many-Valued Logics (((21)) pp. 33-4). My use of Rule 3 avoided the necessity of having a large set of axioms as in Rosser and Turquette, where there would be 25, taking $\sim$, $\supset$ and T_n as basic functions.

2. The 3-valued Lukasiewicz Logic

Although the Lukasiewicz matrices for $\sim$, &, $\vee$, $\longrightarrow$ have been suggested in the literature, I want to give some reasons for their use.

The matrix for $\sim$ is obvious from the discussion about the Russell Paradox.

$\sim$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

Consider the statement '$R \in R$ & $0 \in U$', where 0 is the null class and U the universal class. One would not want to say that it is true because $R \in R$ is not true. Neither would one want to say that it is false because neither $R \in R$ nor $0 \in U$ is false. This only leaves the paradoxical value. This satisfies the property that if one conjunct is true then it can be deleted without altering the value of the conjunction. Now consider the statement, '$R \in R$ & $0 \notin U$'. This is not true because neither $R \in R$ nor $0 \notin U$ is true. Truth and falsity are extremes amongst the significant values because the negation of one is the other. Paradoxicality, being a significant value whose negation is itself, is a neutral, in-between value. Since the conjunction of a false statement with either a true or a false statement is false, the conjunction of a false statement with a paradoxical statement is also false. This now satisfies the property that the conjunction of a false statement with any (significant) statement is false. It is clear that the conjunction of two paradoxical statements is paradoxical. Assuming the commutation of conjunction, the matrix for conjunction is as follows:-

&	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

which is the Lukasiewicz conjunction.

Now consider the statement, $R \in R \vee 0 \in U$. It is certainly not false. If $R \in R$ was replaced by a true or false statement, the truth of $0 \in U$ guarantees the truth of the disjunction. Since, as said earlier, paradoxicality is a neutral value in between truth and falsity, the disjunction, $R \in R \vee 0 \in U$, is true. This now satisfies the property that the disjunction of a true statement with any (significant) statement is true. Now consider the statement, $R \in R \vee 0 \notin U$. It is certainly not true. Since not both disjuncts are false the disjunction cannot be false. Also, since, if $R \in R$ was replaced by a true statement the disjunction would be true and if $R \in R$ was replaced by a false statement the disjunction would be false, $R \in R \vee 0 \notin U$ should be paradoxical on account of paradoxicality being a neutral value in between truth and falsity. This now satisfies the property that if one disjunct is false then it can be deleted without altering the value of the disjunction. It is clear that the disjunction of two paradoxical statements is paradoxical. Assuming the commutation of disjunction, the matrix for disjunction is as follows:-

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

which is the Lukasiewicz disjunction.

The reasons for choosing these matrices are obtained by extending properties of the 2-valued connectives, on the assumption that paradoxicality is a neutral, in-between value. But, there is little else to go on. By an extension of similar properties two implications arise, viz. $\longrightarrow$ and I, given by the matrices:-

$\longrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

I	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1

The Lukasiewicz implications, $\longrightarrow$ and I, satisfy the properties that any (significant) statement implies a true statement, a false statement implies any (significant) statement, if the premiss is true then the value of the implication is the same as that of the conclusion, and if the conclusion is false then the value of the implication is the same as that of the negation of the premiss. In addition, $\longrightarrow$ ensures the validity of $p \longrightarrow p$. Anyway, I is definable in terms of $\sim$ and &, i.e. $p \mathrm{I} q =$ df $\sim (p \,\&\, \sim q)$. $\sim$, &, $\vee$, and I can be used to show the de Morgan Laws.

Using the Wajsberg axiomatisation ((31)) of the 3-valued Lukasiewicz logic. the formal system L is as follows:-

Primitives

1. $p, q, r \ldots.$ (3-valued sentential variables).

2. $\sim, \longrightarrow$ (negation and implication connectives).

Formation Rules

1. A sentential variable is a wff.

2. If A and B are wffs, then $\sim A$ and $A \longrightarrow B$ are wffs.

Definitions

$p \vee q =$ df $(p \longrightarrow q) \longrightarrow q.$

$p \,\&\, q =$ df $\sim (\sim p \vee \sim q).$

$p \longleftrightarrow q =$ df $(p \longrightarrow q) \,\&\, (q \longrightarrow p).$

$p \supset q =$ df $(p \longrightarrow (p \longrightarrow q).$

$p \equiv q =$ df $(p \supset q) \,\&\, (q \supset p).$

$Tp =$ df $\sim (p \longrightarrow \sim p).$

$Fp =$ df $T \sim p.$

$Pp =$ df $\sim Tp \,\&\, \sim Fp.$

$Cp =$ df $Tp \vee Fp.$

$Sp =$ df $Tp \vee Fp \vee Pp.$

The matrix representations of these are as follows:-

$\sim$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1

$\longrightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	1	$\frac{1}{2}$
0	1	1	1

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

&	1	½	0
1	1	½	0
½	½	½	0
0	0	0	0

$\longleftrightarrow$	1	½	0
1	1	½	0
½	½	1	½
0	0	½	1

$\supset$	1	½	0
1	1	½	0
½	1	1	1
0	1	1	1

$\equiv$	1	½	0
1	1	½	0
½	½	1	1
0	0	1	1

T	
1	1
½	0
0	0

F	
1	0
½	0
0	1

P	
1	0
½	1
0	0

C	
1	1
½	0
0	1

Axioms

1. $q \longrightarrow . \ p \longrightarrow q.$
2. $p \longrightarrow q \longrightarrow . \ q \longrightarrow r \longrightarrow . \ p \longrightarrow r.$
3. $((p \longrightarrow \sim p) \longrightarrow p) \longrightarrow p.$
4. $\sim q \longrightarrow \sim p \longrightarrow . \ p \longrightarrow q.$

Rules

1. Substitution for sentential variables.
2. $\vdash A, \ \vdash A \longrightarrow B \Rightarrow \ \vdash B.$

It is shown by Wajsberg ((31)) that this set of axioms yields a complete system; that is, every valid wff is provable. By a similar proof to that of Meta-theorem 2 of the 3-valued significance logic, the Deduction Theorem holds for $\supset$; that is, if $A_1, \text{--}, A_n \vdash_L B$ then $A_1, \text{--}, A_{n-1} \vdash_L A_n \supset B$ (or $A_n \longrightarrow . \ A_n \longrightarrow B$), provided no substitution is made on the variables of A_n. By a similar proof to that of Meta-theorem 3, substitutivity of equivalents holds for $\longleftrightarrow$; that is, if $\vdash_L A \longleftrightarrow B$

then $\vdash_L$ C (A) $\longleftrightarrow$ C (B), with substitution into any argument place. By a similar proof to that of Meta-theorem 4, we have the following result for L:– Let B be a wff of L containing only the connectives, $\supset$, T, &, v, ~T. Let A be a wff of P obtained from B by deleting any T's and replacing ~T by ~. Then, if $\vdash_P$ A then $\vdash_L$ B. By a similar proof to that of Meta-theorem 5, we have the following result:– Let C (p) be a wff of L containing only the connectives, $\supset$, T, &, v, ~T, as in the previous result. Then, if $\vdash_L$ A ≡ B then $\vdash_L$ C (A) ≡ C (B), where A and B can contain any connectives and substitution can be made in C for any argument place.

3. The 4-valued Significance Logic

The connectives of the 4-valued logic are appropriate extensions of the connectives of the two 3-valued logics. The matrix for ~ is clear:–

~	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1
n	n

The matrix for & satisfies the property that if one conjunct is non-significant then the conjunction is non-significant, as in the 3-valued significance logic.

&	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	n
0	0	0	0	n
n	n	n	n	n

The matrix for $\vee$ satisfies the property that if one disjunct is non-significant then this disjunct is ignored when assessing the value of this disjunction, as in 3-valued significance logic.

$\vee$	1	$\frac{1}{2}$	0	n
1	1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0	0
n	1	$\frac{1}{2}$	0	n

The operators T, F, S, P, C satisfy the property that if a sentence is true, false, significant, paradoxical, or true-or-false, respectively, then that sentence operated upon by the respective operator is true, and otherwise false.

T		F		S		P		C	
1	1	1	0	1	1	1	0	1	1
$\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	1	$\frac{1}{2}$	0
0	0	0	1	0	1	0	0	0	1
n	0	n	0	n	0	n	0	n	0

The connectives needed for restricting the variables of the quantifiers, A and S, where these quantifiers are extensions of the two 3-valued ones and reduce over a finite domain to the repeated use of the conjunction & and disjunction $\vee$ above, are $\supset$ and T_n, which are as follows:-

$\supset$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	1	1	1	1
0	1	1	1	1
n	1	1	1	1

T_n	
1	1
$\frac{1}{2}$	n
0	n
n	n

There are essentially two ways of extending the Lukasiewicz $\longrightarrow$. One of these is where a non-significant component causes the implication to be non-significant. This is represented as $\overset{\circ}{\longrightarrow}$ with the matrix:-

$\overset{\circ}{\longrightarrow}$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	1	1	$\frac{1}{2}$	n
0	1	1	1	n
n	n	n	n	n

The other other one is where paradoxical sentences implying true or paradoxical sentences and false sentences implying significant sentences is extended so that non-significant sentences imply any sentence. Also it is taken that for a significant sentence to imply a non-significant one is non-significant. This gives the following:-

$\longrightarrow$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	1	1	$\frac{1}{2}$	n
0	1	1	1	n
n	1	1	1	1

These two implications are introduced mainly because their corresponding equivalences $\overset{\circ}{\longleftrightarrow}$ and $\longleftrightarrow$ are useful in the sequel.

$\dot{\leftrightarrow}$	1	½	0	n
1	1	½	0	n
½	½	1	½	n
0	0	½	1	n
n	n	n	n	n

$\leftrightarrow$	1	½	0	n
1	1	½	0	n
½	½	1	½	n
0	0	½	1	n
n	n	n	n	1

$\dot{+}$ and $\dot{v}$ can be defined as in the 3-valued significance logic because they arise when quantifying over a finite range.

$\dot{+}$	1	½	0	n
1	1	½	0	1
½	½	½	0	½
0	0	0	0	0
n	1	½	0	n

$\dot{v}$	1	½	0	n
1	1	1	1	n
½	1	½	½	n
0	1	½	0	n
n	n	n	n	n

I will now show that the connectives of the set $\{\sim, \&, \supset, T_n\}$ are independent. They will not yield a functionally complete system because the 3-valued Łukasiewicz logic is functionally incomplete and the extensions of the connectives to four values does not introduce any additional ½ values. That is, if a function $\Delta(p)$ is formed using only $\sim$, &, $\supset$ and T_n then if p takes the value 1, 0 or n then $\Delta(p)$ cannot take the value ½ because, whenever the value ½ appears in any of these four connectives, it must have already been present at an earlier stage in the evaluation of $\Delta(p)$. Getting back to the independence of the connectives, I will show it for each one in turn.

(i) Let $\Delta(p)$ be defined in terms of &, $\supset$ and T_n only. If p has the value 1 then $\Delta(p)$ has the value 1. Hence $\sim$ is not definable in terms of &, $\supset$ and T_n only.

(ii) For simplicity, I will show that S is not definable in terms of $\sim$, $\supset$ and T_n. This will show that & is not so definable because S can be defined in terms of $\sim$, & and $\supset$ as follows:- $Sp =$ df $\sim(\sim(p \,\&\, \sim(p \supset p)) \supset \sim(p \supset p))$. Let $\Delta(p)$ be defined in terms of $\sim$, $\supset$ and T_n only. If p takes the value $\frac{1}{2}$, let $\Delta(p)$ take the value $\Delta_{\frac{1}{2}}$. If p takes the value n, let $\Delta(p)$ take the value Δ_n. Either $\Delta_{\frac{1}{2}} = \Delta_n = 1$ or 0, or $\Delta_{\frac{1}{2}} = \frac{1}{2}$ or n and $\Delta_n = \frac{1}{2}$ or n. That is, if $\frac{1}{2}$ is replaced by n everywhere in the matrices of $\sim$, $\supset$ and T_n no inconsistency will arise. In the case of S, an inconsistency will arise because, if p takes the value $\frac{1}{2}$, Sp will take the value 1, while, if p takes the value n, Sp will take the value 0.

(iii) Let $\Delta(p, q)$ be defined in terms of $\sim$, & and T_n only. If p takes the value n and q takes the value n then $\Delta(p, q)$ takes the value n. Hence $\supset$ is not definable in terms of $\sim$, & and T_n.

(iv) Let $\Delta(p)$ be defined in terms of $\sim$, & and $\supset$ only. If p takes a significant value then $\Delta(p)$ also takes a significant value. Hence T_n is not definable in terms of $\sim$, & and $\supset$ only.

This has shown the independence of the connectives. Now I will give the formal axiomatic system LS for the 4-valued significance logic with the above primitives.

Primitives

1. $p, q, r, \ldots\ldots$ (4-valued sentential variables).

2. $\sim, \&, \supset, T_n$ (connectives).

Formation Rules

1. A sentential variable is a wff.

2. If A and B are wffs then $\sim A$, $A \& B$, $A \supset B$ and $T_n A$ are wffs.

Definitions

$Tp = \mathrm{df} \sim (p \supset \sim (p \supset p))$.

$Fp = \mathrm{df}\ T \sim p$.

$Sp = \mathrm{df}\ F (p \,\&\, \sim (p \supset p))$.

$Pp = \mathrm{df}\ Sp \,\&\, \sim Tp \,\&\, \sim Fp$.

$p \vee q = \mathrm{df}\ (Sp \,\&\, Sq \supset \ \sim (\sim p \,\&\, \sim q)) \,\&\, (Sp \,\&\, \sim Sq \supset p) \,\&$
$(Sq \,\&\, \sim Sp \supset q)$.

$Cp = \mathrm{df}\ Tp \vee Fp$.

$p \longrightarrow q = \mathrm{df}\ (((Fp \vee Pp) \,\&\, \sim Sq) \supset .\, p \,\&\, q) \,\&\, (Pp \,\&\, Fq \supset .\ p \vee q)$
$\&\ (\sim (((Fp \vee Pp) \,\&\, \sim Sq) \vee (Pp \,\&\, Fq)) \supset .\ p \supset q)$.

$p \equiv q = \mathrm{df}\ (p \supset q) \,\&\, (q \supset p)$.

$p \longleftrightarrow q = \mathrm{df}\ (p \longrightarrow q) \,\&\, (q \longrightarrow p)$.

$p \overset{\circ}{\longrightarrow} q = \mathrm{df}\ (\sim Sp \vee \sim Sq \supset .\ p \,\&\, q) \,\&\, (Sp \,\&\, Sq \supset .\ p \longrightarrow q)$.

$p \overset{\circ}{\longleftrightarrow} q = \mathrm{df}\ (p \overset{\circ}{\longrightarrow} q) \,\&\, (q \overset{\circ}{\longrightarrow} p)$.

$p + q = \mathrm{df} \sim (\sim p \vee \sim q)$.

$p \dot{\vee} q = \mathrm{df} \sim (\sim p \,\&\, \sim q)$.

The matrix representations are as follows:-

$\sim$	
1	0
$\frac{1}{2}$	$\frac{1}{2}$
0	1
n	n

$\&$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	n
0	0	0	0	n
n	n	n	n	n

$\supset$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	1	1	1	1
0	1	1	1	1
n	1	1	1	1

T_n	
1	1
$\frac{1}{2}$	n
0	n
n	n

T	
1	1
$\frac{1}{2}$	0
0	0
n	0

F	
1	0
$\frac{1}{2}$	0
0	1
n	0

S	
1	1
$\frac{1}{2}$	1
0	1
n	0

P	
1	0
$\frac{1}{2}$	1
0	0
n	0

C	
1	1
$\frac{1}{2}$	0
0	1
n	0

$\vee$	1	$\frac{1}{2}$	0	n
1	1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0	0
n	1	$\frac{1}{2}$	0	n

$\longrightarrow$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	1	1	$\frac{1}{2}$	n
0	1	1	1	n
n	1	1	1	1

$\dot{\longrightarrow}$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	1	1	$\frac{1}{2}$	n
0	1	1	1	n
n	n	n	n	n

$\equiv$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	$\frac{1}{2}$	1	1	1
0	0	1	1	1
n	n	1	1	1

$\longleftrightarrow$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	n
0	0	$\frac{1}{2}$	1	n
n	n	n	n	1

$\dot{\longleftrightarrow}$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	n
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	n
0	0	$\frac{1}{2}$	1	n
n	n	n	n	n

$+$	1	$\frac{1}{2}$	0	n
1	1	$\frac{1}{2}$	0	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
0	0	0	0	0
n	1	$\frac{1}{2}$	0	n

$\mathring{\vee}$	1	$\frac{1}{2}$	0	n
1	1	1	1	n
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	n
0	1	$\frac{1}{2}$	0	n
n	n	n	n	n

An extension of the 3-valued $\simeq$ could have been added but $\longleftrightarrow$ serves the purpose since $p \simeq q$ has the same matrix as $T\ (p \longleftrightarrow q)$.

<u>Axioms</u>

1. C S p.
2. C T p.

3. $C\,p \supset S\,p.$

4. $F\,p \supset \sim T\,p.$

5. $\sim S\,p \supset \sim S \sim p.$

6. $\sim S\,p \vee \sim S\,q \supset \sim S\,(p \,\&\, q).$

7. $\sim S\,p \supset T\,(p \supset q).$

8. $\sim T\,p \,\&\, \sim S\,q \supset T\,(p \supset q).$

9. $T\,p \,\&\, \sim S\,q \supset \sim S\,(p \supset q).$

10. $T\,p \supset T\,T_n\,p.$

11. $\sim T\,p \supset \sim S\,T_n\,p.$

<u>Rules</u>

1. Substitution for sentential variables.

2. $\vdash_{LS} A,\ \vdash_{LS} A \supset B \Longrightarrow\ \vdash_{LS} B.$

3. $\vdash_{L} A\,(p_1, \text{---}, p_n) \Longrightarrow\ \vdash_{LS}\ S\,B_1 \supset . \text{---} . \supset S\,B_n \supset .\ A\,(B_1, \text{---}, B_n),$

   where $p_1, \text{---}, p_n$ are <u>all</u> the variables in A.

Rule 3 is similar to the Rule 3 of the 3-valued significance logic. Again, L is assumed and theses of LS are constructed using it. Also there are common symbols used in P, S, L and LS, it is clear from the context which system the symbol is being used in and if there is any doubt, the appropriate wff will be prefixed by a P, S, L or LS, accordingly.

<u>Theorems</u>

D.R.1. <u>$\vdash_P A\,(p_1, \text{---}, p_n),\ \vdash_{LS} C\,B_1,\ \vdash_{LS} C\,B_2,\ \text{---}\ \vdash_{LS} C\,B_n \Longrightarrow\ \vdash_{LS} A\,(B_1, \text{---}, B_n)$, where $p_1, \text{---}, p_n$ are all the variables in A</u>.

By the completeness of L and the definitions in L of the connective symbols of A, $\vdash_P A(p_1, \text{--}, p_n) \Longrightarrow \vdash_L C\, p_1 \supset . \text{---} . \supset C\, p_n \supset A(p_1, \text{--}, p_n)$. By Rule 3, $\vdash_{LS} S\, B_1 \supset . S\, B_2 \supset . \text{---} . \supset S\, B_n \supset . C\, B_1 \supset . \text{---} . \supset C\, B_n \supset A(B_1, \text{--}, B_n)$. Since $\vdash_{LS} C\, B_1$, --, $\vdash_{LS} C\, B_n$, by Axiom 3, $\vdash_{LS} S\, B_1$, --, $\vdash_{LS} S\, B_n$ and hence $\vdash_{LS} A(B_1, \text{--}, B_n)$.

1. <u>C F p</u>

   A.2, Defn. F : C F p

2. <u>C C p</u>

   P : C (p ∨ q) --- (1)

   (1), D.R.1 : C (T p ∨ F p) --- (2)

   (2), Defn. C : C C p.

3. <u>C P p</u>

   P : C (p & ~ q & ~ r) --- (1)

   (1), D.R.1 : C (S p & ~ T p & ~ F p) --- (2)

   (2), Defn. P : C P p

4. <u>S T p</u>

   A.2, A.3 : S T p

5. <u>S F p</u>

   A.3, T.1 : S F p

6. <u>S C p</u>

   A.3, T.2 : S C p

7. <u>S S p</u>

   A.3, A.1 : S S p

8. <u>$S\ P\ p$</u>

A.3, T.3 : $S\ P\ p$

9. <u>$T\ p \supset C\ p$</u>

P : $p \supset p \vee q$ —— (1)

(1), D.R.1 : $T\ p \supset T\ p \vee F\ p$ —— (2)

(2), Defn. C : $T\ p \supset C\ p$

10. <u>$T\ p \supset S\ p$</u>

T.9 : $T\ p \supset C\ p$ —— (1)

A.3 : $C\ p \supset S\ p$ —— (2)

(1),(2), P, D.R.1 : T.10

11. <u>$F\ p \supset T \sim p$</u>

P : $p \supset p$ —— (1)

(1), D.R.1 : $T \sim p \supset T \sim p$ —— (2)

(2), Defn. F : $F\ p \supset T \sim p$

12. <u>$T\ p \supset F \sim p$</u>

L : $T\ p \supset F \sim p$ —— (1)

(1), R.3 : $S\ p \supset .\ T\ p \supset F \sim p$ —— (2)

(2), P, D.R.1 : $S\ p\ \&\ T\ p \supset F \sim p$ —— (3)

T.10, P, D.R.1 : $T\ p \supset S\ p\ \&\ T\ p$ —— (4)

(3),(4), P, D.R.1 : $T\ p \supset F \sim p$

13. <u>$P\ p \supset S\ p$</u>

P : $p\ \&\ \sim q\ \&\ \sim r \supset p$ —— (1)

(1), D.R.1 : $S\ p\ \&\ \sim T\ p\ \&\ \sim F\ p \supset S\ p$ —— (2)

(2), Defn. P : $Pp \supset S\ p$

14. $\underline{P\ p \supset P \sim p}$

L : $P\ p \supset P \sim p$ —— (1)

(1), R.3 : $S\ p \supset .\ P\ p \supset P \sim p$ —— (2)

(2), P, D.R.1 : $S\ p\ \&\ P\ p \supset P \sim p$ —— (3)

T.13, P, D.R.1 : $P\ p \supset S\ p\ \&\ P\ p$ —— (4)

(3), (4), P, D.R.1 : $P\ p \supset P \sim p$

15. $\underline{F\ p \supset S\ p}$

A.5, P, D.R.1 : $S \sim p \supset S\ p$ —— (1)

T.10, Defn. F : $F\ p \supset S \sim p$ —— (2)

(1), (2), P, D.R.1 : $F\ p \supset S\ p$

16. $\underline{T\ p\ \&\ T\ q \supset T\ (p\ \&\ q)}$

L : $T\ p\ \&\ T\ q \supset T\ (p\ \&\ q)$ —— (1)

(1), R.3 : $S\ p \supset .\ S\ q \supset .\ T\ p\ \&\ T\ q \supset T\ (p\ \&\ q)$ —— (2)

(2), P, D.R.1 : $(S\ p\ \&\ T\ p)\ \&\ (S\ q\ \&\ T\ q) \supset T\ (p\ \&\ q)$ —— (3)

T.10, P, D.R.1 : $T\ p \supset S\ p\ \&\ T\ p$ —— (4)

T.10, P, D.R.1 : $T\ q \supset S\ q\ \&\ T\ q$ —— (5)

(4), (5), P, D.R.1 : $T\ p\ \&\ T\ q \supset .\ (S\ p\ \&\ T\ p)\ \&\ (S\ q\ \&$

$T\ q)$ —— (6)

(3), (6), P, D.R.1 : $T\ p\ \&\ T\ q \supset T\ (p\ \&\ q)$

17. $\underline{(P\ p\ \&\ T\ q) \vee (P\ p\ \&\ P\ q) \vee (T\ p\ \&\ P\ q) \supset P\ (p\ \&\ q)}$

L : $(P\ p\ \&\ T\ q) \vee (P\ p\ \&\ P\ q) \vee (T\ p\ \&\ P\ q) \supset P\ (p\ \&\ q)$ —— (1)

(1), R.3 : $S\ p \supset .\ S\ q \supset .\ (1)$ —— (2)

(2), P, D.R.1 : $S\ p\ \&\ S\ q\ \&\ [(P\ p\ \&\ T\ q) \vee (P\ p\ \&\ P\ q)$

$\vee (T\ p\ \&\ P\ q)] \supset P\ (p\ \&\ q)$ —— (3)

T.10, T.13, P, D.R.1 : $Pp \,\&\, Tq \supset Sp \,\&\, Sq$ --- (4)

T.13, P, D.R.1 : $Pp \,\&\, Pq \supset Sp \,\&\, Sq$ --- (5)

T.10, T.13, P, D.R.1 : $Tp \,\&\, Pq \supset Sp \,\&\, Sq$ --- (6)

(4), (5), (6), P, D.R.1 : $(Pp \,\&\, Tq) \vee (Pp \,\&\, Pq) \vee$
$(Tp \,\&\, Pq) \supset Sp \,\&\, Sq$ --- (7)

(3), (7), P, D.R.1 : $(Pp \,\&\, Tq) \vee (Pp \,\&\, Pq) \vee (Tp \,\&\, Pq)$
$\supset P\,(p \,\&\, q)$.

18. $\underline{(Fp \,\&\, Sq) \vee (Sp \,\&\, Fq) \supset F\,(p \,\&\, q)}$

L : $(Fp \,\&\, Sq) \vee (Sp \,\&\, Fq) \supset F\,(p \,\&\, q)$ --- (1)

(1), R.3 : $Sp \supset .\ Sq \supset .\ (1)$ --- (2)

(2), P, D.R.1 : $Sp \,\&\, Sq \,\&\, [(Fp \,\&\, Sq) \vee (Sp \,\&\, Fq)]$
$\supset F\,(p \,\&\, q)$ --- (3)

T.15, P, D.R.1 : $Fp \,\&\, Sq \supset Sp \,\&\, Sq$ --- (4)

T.15, P, D.R.1 : $Sp \,\&\, Fq \supset Sp \,\&\, Sq$ --- (5)

(4), (5), P, D.R.1 : $(Fp \,\&\, Sq) \vee (Sp \,\&\, Fq) \supset Sp \,\&\, Sq$
--- (6)

(3), (6), P, D.R.1 : $(Fp \,\&\, Sq) \vee (Sp \,\&\, Fq) \supset F\,(p \,\&\, q)$

19. $\underline{Pp \supset \sim Tp}$

P : $p \,\&\, \sim q \,\&\, \sim r \supset \sim q$ --- (1)

(1), D.R.1 : $Sp \,\&\, \sim Tp \,\&\, \sim Fp \supset \sim Tp$ --- (2)

(2), Defn. P : $Pp \supset \sim Tp$

20. $\underline{\sim Sp \supset \sim Tp}$

T.10, P, D.R.1 : $\sim Sp \supset \sim Tp$

21. $\underline{(P\,p \vee F\,p) \,\&\, S\,q \supset . \, T\,(p \supset q)}$

L : $(P\,p \vee F\,p) \,\&\, S\,q \supset T\,(p \supset q)$ —— (1)

(1), R.3 : $S\,p \supset . \, S\,q \supset . \, (P\,p \vee F\,p) \,\&\, S\,q \supset T\,(p \supset q)$ —— (2)

(2), P, D.R.1 : $S\,p \,\&\, S\,q \,\&\, (P\,p \vee F\,p) \,\&\, S\,q \supset T\,(p \supset q)$ —— (3)

T.13 : $P\,p \supset S\,p$ —— (4)

T.15 : $F\,p \supset S\,p$ —— (5)

(4), (5), P, D.R.1 : $P\,p \vee F\,p \supset S\,p$ —— (6)

(3), (6), P, D.R.1 : $(P\,p \vee F\,p) \,\&\, S\,q \supset T\,(p \supset q)$

22. $\underline{T\,p \,\&\, T\,q \supset T\,(p \supset q)}$

L : $T\,p \,\&\, T\,q \supset T\,(p \supset q)$ —— (1)

(1), R.3 : $S\,p \supset . \, S\,q \supset . \, T\,p \,\&\, T\,q \supset T\,(p \supset q)$ —— (2)

(2), P, D.R.1 : $S\,p \,\&\, S\,q \,\&\, T\,p \,\&\, T\,q \supset T\,(p \supset q)$ —— (3)

T.10 : $T\,p \supset S\,p$ —— (4)

T.10 : $T\,q \supset S\,q$ —— (5)

(4), (5), P, D.R.1 : $T\,p \,\&\, T\,q \supset S\,p \,\&\, S\,q$ —— (6)

(3), (6), P, D.R.1 : $T\,p \,\&\, T\,q \supset T\,(p \supset q)$

23. $\underline{T\,p \,\&\, P\,q \supset P\,(p \supset q)}$

L : $T\,p \,\&\, P\,q \supset P\,(p \supset q)$ —— (1)

(1), R.3 : $S\,p \supset . \, S\,q \supset . \, T\,p \,\&\, P\,q \supset P\,(p \supset q)$ —— (2)

(2), P, D.R.1 : $S\,p \,\&\, S\,q \,\&\, T\,p \,\&\, P\,q \supset P\,(p \supset q)$ —— (3)

T.10 : $T\,p \supset S\,p$ —— (4)

T.13 : $P\,q \supset S\,q$ —— (5)

(3), (4), (5), P, D.R.1 : $T\,p \,\&\, P\,q \supset P\,(p \supset q)$

24. $\underline{T\,p \,\&\, F\,q \supset F\,(p \supset q)}$

L : $T\,p \,\&\, F\,q \supset F\,(p \supset q)$ — (1)

(1), R.3 : $S\,p \supset .\ S\,q \supset .\ T\,p \,\&\, F\,q \supset F\,(p \supset q)$ — (2)

(2), P, D.R.1 : $S\,p \,\&\, S\,q \,\&\, T\,p \,\&\, F\,q \supset F\,(p \supset q)$ — (3)

T.10, T.15, (3), P, D.R.1 : $T\,p \,\&\, F\,q \supset F\,(p \supset q)$

25. $\underline{T\,p \vee F\,p \vee P\,p \vee \sim S\,p}$

P : $p \vee \sim p$ — (1)

(1), D.R.1 : $(T\,p \vee F\,p \vee \sim S\,p) \vee \sim (T\,p \vee F\,p \vee \sim S\,p)$

— (2)

(2), P, D.R.1 : $T\,p \vee F\,p \vee \sim S\,p \vee (\sim T\,p \,\&\, \sim F\,p \,\&\, S\,p)$

— (3)

(3), Defn. P : $T\,p \vee F\,p \vee P\,p \vee \sim S\,p$

26. $\underline{(T\,p \supset T\,q) \,\&\, (F\,p \supset T\,q) \,\&\, (P\,p \supset T\,q) \,\&\, (\sim S\,p \supset T\,q)}$

$\underline{\supset T\,q}$

P : $(p \supset t) \,\&\, (q \supset t) \,\&\, (r \supset t) \,\&\, (s \supset t) \supset (p \vee q \vee r$

$\vee\, s \supset t)$ — (1)

(1), D.R.1 : $(T\,p \supset T\,q) \,\&\, (F\,p \supset T\,q) \,\&\, (P\,p \supset T\,q) \,\&$

$(\sim S\,p \supset T\,q) \supset (T\,p \vee F\,p \vee P\,p \vee \sim S\,p \supset T\,q)$ — (2)

(2), T.25, P, D.R.1 : $(T\,p \supset T\,q) \,\&\, (F\,p \supset T\,q) \,\&\, (P\,p \supset$

$T\,q) \,\&\, (\sim S\,p \supset T\,q) \supset T\,q$

D.R.2. $\underline{\vdash_{LS} T\,A \Longrightarrow \vdash_{LS} A}$

T.9 : $C\,A$ — (1)

P : $T\,p \supset p$ — (2)

(1), (2), D.R.1 : $T\,A \supset A$ --- (3)

(3) : A

Now I can give the proof of completeness for the above set of axioms and rules with respect to the 4-valued matrices. But first I will prove a lemma.

Lemma

Let B be a wff of LS and let $p_1, \text{---}, p_n$ be the distinct variables occurring in B. Let A_i be Tp_i, Pp_i, Fp_i or $\sim Sp_i$ according as the value a_i of p_i is 1, $\frac{1}{2}$, 0 or n. Let B' be TB, PB, FB or $\sim SB$ according as the value of B, for values a_1, a_2, ---, a_n of p_1, p_2, ---, p_n is 1, $\frac{1}{2}$, 0 or n. Then $\vdash_{LS} A_1 \,\&\, \text{---} \,\&\, A_n \supset B'$.

Proof

By induction on the number of variable occurrences.

It is trivial in the case of a single variable.

(i) B is $\sim B_1$

By ind. hyp., $A_1 \,\&\, \text{---} \,\&\, A_n \supset B_1'$. If B_1 has the value 1, $\frac{1}{2}$, 0 or n, then B has the value 0, $\frac{1}{2}$, 1 or n, respectively. So, if B_1' is TB_1, PB_1, FB_1 or $\sim SB_1$, then B' is $F \sim B_1$, $P \sim B_1$, $T \sim B_1$ or $\sim S \sim B_1$, respectively.

By T.12, T.14, T.11 and A.5, $TB_1 \supset F \sim B_1$, $PB_1 \supset P \sim B_1$, $FB_1 \supset T \sim B_1$, and $\sim SB_1 \supset \sim S \sim B_1$. Hence $B_1' \supset B'$ and, by using P and D.R.1, $A_1 \,\&\, \text{---} \,\&\, A_n \supset B'$.

(ii) <u>B is B_1 & B_2</u>

By ind. hyp., A_1 & — & $A_n \supset B_1'$ and A_1 & — & $A_n \supset B_2'$, where p_1,—,p_n are all the distinct variables in B_1 & B_2.

(a) If B_1 has the value 1 and B_2 has the value 1, then B has the value 1. So, if B_1' is TB_1 and B_2' is TB_2, then B' is T (B_1 & B_2).

By T.16, T p & T q $\supset$ T (p & q) and hence TB_1 & TB_2 $\supset$ T (B_1 & B_2) and B_1' & B_2' $\supset B'$. By using P and D.R.1, A_1 & — & $A_n \supset B'$.

(b) If B_1 has the value $\frac{1}{2}$ and B_2 has the value 1 or $\frac{1}{2}$ or if B_1 has the value 1 and B_2 has the value $\frac{1}{2}$, then B has the value $\frac{1}{2}$. So, if B_1' is PB_1 and B_2' is TB_2 or PB_2 or if B_1' is TB_1 and B_2' is PB_2, then B' is P (B_1 & B_2).

By T.17, (PB_1 & TB_2) v (PB_1 & PB_2) v (TB_1 & PB_2) $\supset$ P (B_1 & B_2). By P and D.R.1, PB_1 & TB_2 $\supset$ P (B_1 & B_2), PB_1 & PB_2 $\supset$ P (B_1 & B_2) and TB_1 & PB_2 $\supset$ P (B_1 & B_2). Hence B_1' & B_2' $\supset B'$ in all three cases. By using P and D.R.1, A_1 & —— & $A_n \supset B'$.

(c) If B_1 has the value 0 and B_2 has the value 1, $\frac{1}{2}$ or 0 or if B_1 has the value 1 or $\frac{1}{2}$ and B_2 has the value 0, then B has the value 0. So, if B_1' is FB_1 and B_2' is TB_2, PB_2 or FB_2 or if B_1' is TB_1 or PB_1 and B_2' is FB_2 then B' is F (B_1 & B_2).

By T.18, (FB_1 & SB_2) v (SB_1 & FB_2) $\supset$ F (B_1 & B_2).

By T.10, T.13 and T.15 and by using P and D.R.1, $FB_1 \ \& \ TB_2 \supset F\ (B_1 \ \& \ B_2)$, $FB_1 \ \& \ PB_2 \supset F\ (B_1 \ \& \ B_2)$, $FB_1 \ \& \ FB_2 \supset F\ (B_1 \ \& \ B_2)$, $TB_1 \ \& \ FB_2 \supset F\ (B_1 \ \& \ B_2)$ and $PB_1 \ \& \ FB_2 \supset F\ (B_1 \ \& \ B_2)$. Hence in all cases $B_1' \ \& \ B_2' \supset B'$ and, by using P and D.R.1, $A_1 \ \& \ \text{--} \ \& \ A_n \supset B'$.

(d) If B_1 has the value n or if B_2 has the value n then B has the value n. So, if B_1' is $\sim SB_1$ or if B_1' is TB_1, PB_1 or FB_1 and B_2' is $\sim SB_2$ then B' is $\sim S\ (B_1 \ \& \ B_2)$.

By A.6, $\sim SB_1 \vee \sim SB_2 \supset \sim S\ (B_1 \ \& \ B_2)$.

By P and D.R.1, $\sim SB_1 \supset \sim S\ (B_1 \ \& \ B_2)$ and $\sim SB_2 \supset \sim S\ (B_1 \ \& \ B_2)$. In each case, either B_1' is $\sim SB_1$ or B_2' is $\sim SB_2$ and hence $B_1' \supset B'$ or $B_2' \supset B'$. Hence $A_1 \ \& \ \text{--} \ \& \ A_n \supset B'$.

(iii) <u>B is $B_1 \supset B_2$</u>

By ind. hyp., $A_1 \ \& \ \text{--} \ \& \ A_n \supset B_1'$ and $A_1 \ \& \ \text{--} \ \& \ A_n \supset B_2'$ where $p_1, \text{--}, p_n$ are all the distinct variables in $B_1 \ \& \ B_2$.

(a) If B_1 has the value $\frac{1}{2}$, 0 or n then B has the value 1. So, if B_1' is PB_1, FB_1 or $\sim SB_1$ then B is $T\ (B_1 \supset B_2)$.

By T.21, A.8, T.19, and by using P and D.R.1, $PB_1 \supset T\ (B_1 \supset B_2)$. By using A.4 as well, $FB_1 \supset T\ (B_1 \supset B_2)$. By A.7, $\sim SB_1 \supset T\ (B_1 \supset B_2)$. Hence $B_1' \supset B'$ and $A_1 \ \& \ \text{--} \ \& \ A_n \supset B'$.

(b) If B_1 has the value 1 and B_2 has the value 1, $\frac{1}{2}$, 0 or n then B has the value 1, $\frac{1}{2}$, 0 or n, respectively. So, if

B_1' is TB_1 and B_2' is TB_2, PB_2, FB_2 or $\sim SB_2$ then B' is $T\ (B_1 \supset B_2)$, $P\ (B_1 \supset B_2)$, $F\ (B_1 \supset B_2)$ or $\sim S\ (B_1 \supset B_2)$, respectively.

By T.22, T.23, T.24 and A.9, $B_1'\ \&\ B_2' \supset B'$ and hence $A_1\ \&\ \text{--}\ \&\ A_n \supset B'$.

(iv) <u>B is $T_n B_1$</u>

If B_1 takes the value 1 then B takes the value 1 and if B_1 takes the value $\frac{1}{2}$, 0 or n then B takes the value n. So, if B_1' is TB_1 then B' is $T\ T_n\ B_1$ and if B_1' is PB_1, FB_1 or $\sim SB_1$ then B' is $\sim S\ T_n\ B_1$.

By A.10, A.11, A.4, T.19 and T.20, and using P and D.R.1, $B_1' \supset B'$ and hence $A_1\ \&\ \text{--}\ \&\ A_n \supset B'$.

<u>Meta-theorem 6</u>

If B is valid according to the 4-valued matrices, then B is a thesis of the axiomatic system LS.

<u>Proof</u>

Let $p_1, \text{--}, p_n$ be the distinct variables of B. Let $A_1, \text{--}, A_n$ be as in the lemma. Since B is valid, B' is TB, independently of the values of $p_1, \text{--}, p_n$. Hence $\vdash_{LS} A_1\ \&\ \text{--}\ \&\ A_n \supset TB$, irrespective of whether p_n has the value 1, $\frac{1}{2}$, 0 or n. Therefore, $\vdash_{LS} A_1\ \&\ \text{--}\ \&\ A_{n-1}\ \&\ Tp_n \supset TB$. By P and D.R.1, $\vdash_{LS} A_1\ \&\ \text{--}\ \&\ A_{n-1} \supset (Tp_n \supset TB)$. Similarly, $\vdash_{LS} A_1\ \&\ \text{--}\ \&\ A_{n-1} \supset (Pp_n \supset TB)$, $\vdash_{LS} A_1\ \&\ \text{--}\ \&$

$A_{n-1} \supset (Fp_n \supset TB)$ and $\vdash_{LS} A_1 \mathbin{\&} \text{---} \mathbin{\&} A_{n-1} \supset (\sim Sp_n \supset TB)$. By P and D.R.1, $\vdash_{LS} A_1 \mathbin{\&} \text{---} \mathbin{\&} A_{n-1} \supset . (Tp_n \supset TB) \mathbin{\&} (Pp_n \supset TB) \mathbin{\&} (Fp_n \supset TB) \mathbin{\&} (\sim Sp_n \supset TB)$. Using T.26, $\vdash_{LS} A_1 \mathbin{\&} \text{---} \mathbin{\&} A_{n-1} \supset TB$. By repeating this procedure for each A_i, $1 \leqslant i \leqslant n-1$, $\vdash_{LS} TB$. Hence, by D.R.2, $\vdash_{LS} B$.

<u>Meta-theorem 7</u>

The Deduction Theorem holds in LS for '$\supset$', i.e. if $A_1, \text{---}, A_n \vdash_{LS} B$ then $A_1, \text{---}, A_{n-1} \vdash_{LS} A_n \supset B$, provided no substitution is made on variables of A_n.

<u>Proof</u>

The proof is similar to that of Meta-theorem 2. Since $p \supset p$, $p \supset (q \supset p)$ and $(p \supset q) \supset (p \supset . q \supset r) \supset . p \supset r$ are all valid by the 4-valued matrices, the proof is clear.

<u>Meta-theorem 8</u>

Substitutivity of Equivalents holds in LS for $\Longleftrightarrow$, i.e. if $\vdash_{LS} A \Longleftrightarrow B$ then $\vdash_{LS} C(A) \Longleftrightarrow C(B)$, where substitution can be made in C for any argument place.

<u>Proof</u>

By Theorem 6, $\vdash_{LS} A \Longleftrightarrow B \supset . \sim A \Longleftrightarrow \sim B$, $\vdash_{LS} A \Longleftrightarrow B \supset . D \mathbin{\&} A \Longleftrightarrow D \mathbin{\&} B$, $\vdash_{LS} A \Longleftrightarrow B \supset . A \mathbin{\&} D \Longleftrightarrow B \mathbin{\&} D$, $\vdash_{LS} A \Longleftrightarrow B \supset . D \supset A \Longleftrightarrow D \supset B$, $\vdash_{LS} A \Longleftrightarrow B \supset . A \supset D \Longleftrightarrow B \supset D$, $\vdash_{LS} A \Longleftrightarrow B$

$\supset . T_n A \Leftrightarrow T_n B$. By applying these to each connective in turn, $\vdash_{LS} C(A) \Leftrightarrow C(B)$.

Meta-theorem 9

Let B be a wff of LS containing only the connectives, $\supset$, T_n, T, &, v, $\sim$T. Let A be a wff of P obtained from B by deleting any T's or T_n's and replacing $\sim$T by $\sim$. Then, if $\vdash$ A then $\vdash_{LS}$ B.

Proof

The proof is similar to that of Meta-theorem 6, except that there are only two values, 1 and not-1. The lemma can be stated as follows: Let B be a wff of LS and let $p_1, \text{--}, p_n$ be the distinct variables occurring in B. Let A_i be Tp_i or $\sim Tp_i$ according as the value a_i of p_i is 1 or not-1. Let B' be TB or $\sim$TB according as the value of B, for values $a_1, \text{--}, a_n$ of $p_1, \text{--}, p_n$, is 1 or not-1. Then $\vdash_{LS} A_1 \ \& \text{ -- } \& \ A_n \supset B'$. This is proved similarly to the lemma to Meta-theorem 6 making use of Meta-theorem 6 in the process.

One can prove $\vdash_{LS} A_1 \ \& \text{ -- } \& \ A_{n-1} \supset . Tp_n \supset TB$ and $\vdash_{LS} A_1 \ \& \text{ -- } \& \ A_{n-1} \supset . \sim Tp_n \supset TB$. By using Meta-theorem 6, $\vdash_{LS} A_1 \ \& \text{ -- } \& \ A_{n-1} \supset TB$ and by repetition $\vdash_{LS}$ TB. Hence $\vdash_{LS}$ B.

Meta-theorem 10

Let C(p) be a wff of LS containing only the connectives, $\supset$, T_n, T, &, v, ~T, as in Theorem 9. Then, if $\vdash_{LS}$ A ≡ B then $\vdash_{LS}$ C (A) ≡ C (B), where substitution is similar to Theorem 8. (A and B can contain any connectives.)

Proof

By Theorem 6, $\vdash_{LS} A \equiv B \supset . D \supset A \equiv D \supset B$, $\vdash_{LS} A \equiv B \supset . A \supset D \equiv B \supset D$, $\vdash_{LS} A \equiv B \supset . T_n A \equiv T_n B$, $\vdash_{LS} A \equiv B \supset . TA \equiv TB$, $\vdash_{LS} A \equiv B \supset . A \,\&\, D \equiv B \,\&\, D$, $\vdash_{LS} A \equiv B \supset . D \,\&\, A \equiv D \,\&\, B$, $\vdash_{LS} A \equiv B \supset . A \vee D \equiv B \vee D$, $\vdash_{LS} A \equiv B \supset . D \vee A \equiv D \vee B$, $\vdash_{LS} A \equiv B \supset . \sim TA \equiv \sim TB$. By applying these to each connective of A in turn, $\vdash_{LS}$ C (A) ≡ C (B).

. . .

This completes the account of the 4-valued significance logic.

In each of the three logics developed, only sufficiently many theorems for the completeness proof have been derived. One must use the matrices to test for the validity of particular wffs. Meta-theorems 5 and 10 are put in just to aid one in testing for the validity of certain types of wffs. It is clear in each of the three formal systems that all axioms are valid and the rules preserve validity, and hence a wff is valid iff it is a thesis.

CHAPTER II

PREDICATE LOGIC

In this chapter, I will extend the three sentential logics developed in the first chapter so as to include predicates and subjects.

1. The 3-valued Significance Logic

The quantifiers A and S were introduced in the last chapter. A satisfies the property that if $B(x)$ is non-significant for some x then $(Ax)B(x)$ is non-significant, if $B(x)$ is true for all x then $(Ax)B(x)$ is true, and if $B(x)$ is significant for all x and false for some x then $(Ax)B(x)$ is false. S satisfies the property that if $B(x)$ is non-significant for all x then $(Sx)B(x)$ is non-significant; if $B(x)$ is true for some x then $(Sx)B(x)$ is true; and if $B(x)$ is not true, for all x, and false for some x then $(Sx)B(x)$ is false. The quantifiers $\forall$ and $\exists$ can then be defined in terms of A and S.

Before giving the formal axiomatic system for the 3-valued significance predicate logic, I will present the 2-valued predicate logic which, being an extension of the 2-valued sentential logic, I will call P.

Add the following to the sentential system P:-

Primitives

3. x, y, z, ... (subject variables).

4. f, g, h, ... (predicate variables).

5. A (universal quantifier).

Formation Rules

3. If f is an n-ary predicate variable and $x_1, \text{---}, x_n$ are subject variables then $f(x_1, \text{---}, x_n)$ is a wff.

4. If B is a wff and x is a subject variable then (Ax)B is a wff.

Definition

9. $(Sx)A = \text{df} \sim (Ax) \sim A$.

Axiom

4. $(Ax)A \supset A$.

(Axioms 1 to 3 of the sentential logic are written in schematic form.)

Rules

3. Substitution for subject variables, free and bound, with the provisos in [2], p192.

4. $\vdash_p A \supset B \Longrightarrow \vdash_p A \supset (Ax)B$, where x is not free in A.

This 2-valued system P is complete.

The 3-valued system S is obtained by adding the following to the sentential system S:-

Primitives

3. x, y, z, ... (subject variables).

4. f, g, h, ... (predicate variables).

5. A (universal quantifier), S (existential quantifier).

Formation Rules

3. If f is an n-ary predicate variable and $x_1, \text{---}, x_n$ are subject variables then $f(x_1, \text{--}, x_n)$ is a wff.

4. If A is a wff and x is a subject variable then (Ax)A and (Sx)A are wffs.

Definitions

$(\mathrm{E}x)A = \mathrm{df} \sim (\mathrm{A}x) \sim A$

$(\forall x)A = \mathrm{df} \sim (\mathrm{S}x) \sim A.$

Axioms (all the sentential axioms are written in schematic form)

9. $(\mathrm{S}x) \sim \mathrm{S}\ A(x) \supset \sim \mathrm{S}\ (\mathrm{A}x)\ A(x).$

10. $(\mathrm{S}x)\ \mathrm{T}\ A(x) \supset \mathrm{T}\ (\mathrm{S}x)\ A(x).$

11. $(\mathrm{A}x) \sim \mathrm{T}\ A(x)\ \&\ (\mathrm{S}x)\ \mathrm{F}\ A(x) \supset \mathrm{F}\ (\mathrm{S}x)\ A(x).$

12. $(\mathrm{A}x) \sim \mathrm{S}\ A(x) \supset \sim \mathrm{S}\ (\mathrm{S}x)\ A(x).$

13. $(\mathrm{A}x)\ A \supset A.$

14. $A \supset (\mathrm{S}x)\ A.$

Rules

4. Substitution for subject variables, free and bound, with the provisos in [2], p192.

5. $\vdash_s A \supset B \Longrightarrow \vdash A \supset (\mathrm{A}x)\ B$, where x is not free in A.

6. $\vdash_P \mathcal{A}(A_1(x_{1,1},\ldots,x_{1,i_1}),\ldots,A_n(x_{n,1},\ldots,x_{n,i_n})) \Rightarrow \vdash_S (\Lambda x_{1,1},\ldots,x_{1,i_1}) S \mathcal{B}_1(x_{1,1},\ldots,x_{1,i_1}) \supset. \ldots \supset (\Lambda x_{n,1},\ldots,x_{n,i_n}) S \mathcal{B}_n(x_{n,1},\ldots,x_{n,i_n}) \supset \mathcal{A}(\mathcal{B}_1(x_{1,1},\ldots,x_{1,i_1}),\ldots,\mathcal{B}_n(x_{n,1},\ldots,x_{n,i_n}))$, where $A_1,\ldots,A_n$ are the only wff-schemata in $\mathcal{A}$ and $x_{j,1},\ldots,x_{j,i_j}$ are the only variables (i.e. free or bound by $\mathcal{A}$) in A_j and the only free variables (i.e. free or bound by $\mathcal{A}$) in $\mathcal{B}_j$. [The $\mathcal{A}$ and $\mathcal{B}_j$'s are schemata for the wff-schemata A, B, etc.]

Rule 6 is similar to Rule 3 of S but it caters for wffs of predicate logic. There is a similar relation between the predicate systems P, S, L, and LS as there is between the sentential systems P, S, L and LS. Again common symbols have been employed in these systems.

Theorems

D.R.1. $\vdash_S A \Rightarrow \vdash_S (\Lambda x)\, A.$

R.S: $\vdash_S (p \supset p) \supset A \Rightarrow \vdash_S (p \supset p) \supset (\Lambda x)\, A$ __(1)

S: $(p \supset p) \supset A$ __ (2).

(1),(2): $(p \supset p) \supset (\Lambda x)\, A$ __ (3)

(3), S: $(\Lambda x)\, A.$

D.R.2. $\vdash_S A \supset B \Rightarrow \vdash_S (\Lambda x)\, A \supset (\Lambda x)\, B.$

A.13, S: $A \supset B \supset. (\Lambda x) A \supset B$ __(1)

(1) : $(\Lambda x)\, A \supset B$ __(2)

(2) , R.5 : $(\Lambda x)\, A \supset (\Lambda x)\, B.$

1. $(\Lambda x)\, A \supset (Sx)\, A.$

A.13, A.14, S: $(\Lambda x)\, A \supset (Sx)\, A.$

2. $(\Lambda x)TA(x) \supset T(\Lambda x)A(x).$

P: $(\Lambda x)TA(x) \supset T(\Lambda x)A(x)$ __(1)

(1), R.6: $(\Lambda x)\, S\, A(x) \supset. (\Lambda x)TA(x) \supset T(\Lambda x)A(x)$ __(2)

S : $(p \supset. q \supset r) \supset (q \,\&\, p \supset r)$ __(3)

(2),(3),R.2: $(\Lambda x)\ S\ A(x)\ \&\ (\Lambda x)\ TA(x) \supset T(\Lambda x)\ A(x)$ ___(4)

S : $TA(x) \supset S\ A(x)$ ___(5)

(5), D.R.2: $(\Lambda x)\ TA(x) \supset (\Lambda x)\ S\ A(x)$ ___(6)

S : $p \supset q\ .\supset.\ p \supset q\ \&\ p$ ___(7)

(6),(7), R.2: $(\Lambda x)\ TA(x) \supset (\Lambda x)\ S\ A(x)\ \&\ (\Lambda x)\ TA(x)$ ___(8)

(4),(8),S: $(\Lambda x)\ T\ A(x) \supset T(\Lambda x)\ A(x)$.

3. <u>$(Sx)\ F\ A(x)\ \&\ (\Lambda x)\ S\ A(x) \supset F\ (\Lambda x) A\ (x)$.</u>

P: $(Sx)\ F\ A\ (x)\ \&\ \Lambda x)\ S\ A(x) \supset F\ (\Lambda x)\ A(x)$ ___(1)

(1), R.6: $(\Lambda x)\ SA(x) \supset.\ (Sx)\ F\ A(x)\ \&\ (\Lambda x)\ S\ A(x)$
$\supset F\ (\Lambda x)\ A\ (x)$ ___(2)

S: $(p \supset.\ q\ \&\ p \supset r) \supset (q\ \&\ p \supset r)$ ___(3)

(2),(3), R.2: $(Sx)\ F\ A\ (x)\ \&\ (\Lambda x)\ S\ A\ (x) \supset F\ (\Lambda x)\ A(x)$.

4. <u>$T(Sx)\ A(x) \supset S(x)\ T\ A(x)$.</u>

S: $T(Sx)\ A(x) \supset \sim F\ (Sx)\ A(x)$ ___(1)

P: $((\Lambda x)\ A(x)\ \&\ (Sx)B(x) \supset C) \supset (\sim C \supset \sim (\Lambda x)A(x)$
$v \sim (Sx)B(x))$ ___(2)

S, D.R.1: $(\Lambda x)\ S \sim T\ A(x)$ ___(3)

S, D.R.1: $(\Lambda x)\ S\ F\ A\ (x)$ ___(4)

S:S $F\ (Sx)\ A(x)$ ___(5)

(2),(3),(4),(5), R.6: $((\Lambda x) \sim TA(x)\ \&\ (Sx)\ FA(x) \supset F\ (Sx)\ A(x))$
$\supset (\sim F(Sx)\ A(x) \supset \sim (\Lambda x) \sim T\ A(x)\ v \sim (Sx)\ F\ A(x))$
$)$ — (6)

A.11 (6): $\sim F\ (Sx)\ A(x) \supset \sim (\Lambda x) \sim T\ A\ (x)\ v \sim (Sx)\ F\ A(x)$ ___(7)

(1),(7),S: $T\ (Sx)\ A\ (x) \supset \sim (\Lambda x) \sim T\ A\ (x)\ v \sim (Sx)\ F\ A(x)$ ___(8)

P: $\sim (\Lambda x) \sim A(x) \supset (Sx)\ A(x)$ ___(9)

S,D.R.1: $(\Lambda x)\ S\ T\ A(x)$ ___(10)

(9), (10), R 6: $\sim(\Lambda x) \sim T\ A(x) \supset (Sx)\ T\ A\ (x)$ ___(11)

(8), (11), S: $T\ (Sx)\ A(x) \supset (Sx)\ T\ A(x)\ v \sim (Sx)\ F\ A(x)$ ___(12)

S: $T\ (Sx)\ A(x) \supset S\ (Sx)\ A(x)$ ___(13)

P: $((Ax) \sim A(x) \supset \sim B(x)) \supset (B(x) \supset (Sx)\ A(x))$ ___(14)

S, D.R.1: $(Ax)\ SS\ A(x)$ ___(15)

S: $S\ S\ (Sx)\ A(x)$ ___(16)

(14),(15),(16), R.6: $((Ax) \sim S\ A(x) \supset \sim S(Sx)\ A(x))$
$\supset (S\ (Sx)\ A(x) \supset (Sx)\ S\ A(x))$ ___(17)

A.12, (17): $S(Sx)\ A(x) \supset (Sx)\ S\ A(x)$ ___(18)

(13), (18), S: $T(Sx)\ A(x) \supset (Sx)\ S\ A(x)$ ___(19)

S, Defn. S: $(Sx)\ S\ A(x) \supset (Sx)\ (T\ A(x)\ v\ F\ A(x))$ ___(20)

P: $(Sx)\ (A(x)\ v\ B(x)) \supset (Sx)\ A(x)\ v\ (Sx)\ B(x)$ ___(21)

S, D.R.1: $(Ax)\ S\ T\ A(x)$ ___(22)

S, D.R.1: $(Ax)\ S\ F\ A(x)$ ___(23)

(21),(22), (23), R.6: $(Sx)(TA(x)\ v\ F\ A(x)) \supset (Sx)\ T\ A(x)$
$v\ (Sx)\ F\ A(x)$ ___(24)

(19),(20), (24), S: $T(Sx)\ A(x) \supset (Sx)\ T\ A(x)\ v\ (Sx)\ F\ A(x)$ ___(25)

(12), (25), R.6.: $T\ (Sx)\ A(x) \supset (Sx)\ T\ A(x)$.

5. <u>$S\ (Sx)\ A(x) \supset (Sx)\ S\ A(x)$.</u>

(18) of T4: $S\ (Sx)\ A(x) \supset (Sx)\ S\ A(x)$.

6. <u>$S\ (Ax)\ A(x) \supset (Ax)\ S\ A(x)$.</u>

P: $((Sx) \sim A(x) \supset \sim B(x)) \supset (B(x) \supset (Ax)\ A(x))$ ___(1)

S, D.R.1: $(Ax)\ SS\ A(x)$ ___(2)

S: $S\ S\ (Ax)\ A(x)$ ___(3)

(1), (2), (3), R.6: $((Sx) \sim S\ A(x) \supset \sim S(Ax)\ A(x))$
$\supset (S\ (Ax)\ A(x) \supset (Ax)\ S\ A(x))$ ___(4)

A.9., (4) : $S(Ax)\ A(x) \supset (Ax)\ S\ A(x)$.

The completeness proof for the axioms and rules with respect to the properties that Λ and S satisfy can now be given. Firstly, I prove the Deduction Theorem for $\supset$.

Meta-theorem 1.

The Deduction Theorem holds in S for $\supset$, i.e. if $A_1, \ldots, A_n \vdash_S B$ then $A_1, \ldots, A_{n-1} \vdash_S A_n \supset B$, provided Rule 5 is not used to generalise on any variable of A_n and Rule 4 is not used to change a free variable of A_n.

Proof

This proceeds the same way as Meta-theorem 2 of Chapter 1, but, account must be taken of Rules 4, 5 and 6. If $\vdash_S A_n \supset C(x)$ then by Rule 4, $\vdash_S A_n \supset C(y)$, where x and y are free variables, with the proviso in [2], p192. If $\vdash_S A_n \supset C((Qx)D(x))$ then also by Rule 4, $\vdash_S A_n \supset C((Qy)D(y))$, where Q is Λ or S, with the proviso in [2], p192. If $\vdash_S A_n \supset . A \supset B$ then by the sentential S, $\vdash_S A_n \& A \supset B$. Since x is not free in $A_n \& A$, $\vdash_S A_n \& A \supset (\Lambda x) B$, by Rule 5. By the sentential S, $\vdash_S A_n \supset . A \supset (\Lambda x)B$.
If, by Rule 6, $\vdash_S (\Lambda x_{1,1}, \ldots, x_{1,i_1})\, S\mathcal{B}_1 (x_{1,1}, \ldots, x_{1,i_1}) \supset \ldots\ldots \supset (\Lambda x_{n,1}, \ldots, x_{n,i_n})\, S\mathcal{B}_n(x_{n,1}, \ldots, x_{n,i_n}) \supset \mathcal{A}(\mathcal{B}_1 (x_{1,1}, \ldots, x_{1,i_1}), \ldots, \mathcal{B}_n(x_{n,1}, \ldots, x_{n,i_n}))$, for some schemata $\mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_n$, then $\vdash_S A_n \supset$ the above expression, since $\vdash_S p \supset (q \supset p)$.
Hence the Deduction Theorem follows.

Meta-theorem 2.

If B is valid according to the 3-valued matrices and the properties stated for Λ and S, then B is a thesis of the axiomatic system S.

Proof.

The proof is modelled on the proof for the completeness of the 2-valued predicate calculus given in Church, [2], p.238-245.

As in Church, I will introduce some definitions. If Γ is any class of wffs and B is any wff, then $\Gamma \vdash B$ if there is a finite number of wffs $A_1, \ldots, A_n$ of Γ such that $A_1, \ldots, A_n \vdash B$. A class Γ of wffs is called inconsistent if there exists a wff B such that $\Gamma \vdash B$ and $\Gamma \vdash \sim B$. If no such B exists, then Γ is consistent. If Γ is any class of wffs and C is any wff, then C is consistent with Γ if the class $\{C\} \cup \Gamma$ is consistent otherwise, C is inconsistent with Γ. A class Γ of wffs is called a maximal consistent class if Γ is consistent and if C is con sistent with Γ then $C \in \Gamma$.

Lemma.

Every consistent class Γ of wffs can be extended to a maximal consistent class $\bar{\Gamma}$, ie., there exists a maximal consistent class $\bar{\Gamma}$ containing Γ.

Proof.

Enumerate the wffs of the system. Given any class Γ of wffs, define an infinite sequence of classes $\Gamma^0, \Gamma^1, \Gamma^2, \ldots,$ as follows: Γ^0 is Γ. If the $(n + 1)$ -st wff is consistent with Γ^n, then $\Gamma^{n+1} = \Gamma^n \cup \{(n+1)$ -st wff$\}$. Otherwise Γ^{n+1} is Γ^n. $\Gamma^0, \Gamma^1, \ldots,$ are consistent classes of wffs. Let $\bar{\Gamma}$ be the union of the classes $\Gamma^0, \Gamma^1, \ldots,$. If Γ is consistent then $\bar{\Gamma}$ is consistent, for, if $\bar{\Gamma}$ is inconsistent then $A_1, A_2, \ldots, A_n \vdash B$ and $A_1, A_2, \ldots, A_m \vdash \sim B$, for some wff B of $\bar{\Gamma}$, where $A_1, \ldots, A_m$ are members of Γ^a, where a is the greatest number assigned to any of the wffs $A_1, \ldots, A_m$ in the enumeration. Since Γ^a is consistent, so is $\bar{\Gamma}$. $\bar{\Gamma}$ is also a maximal class if Γ is consistent. Let C be consistent with $\bar{\Gamma}$. If C is the (n+1)-st wff then C is consistent with Γ^n and $C \in \Gamma^{n+1}$. Hence $C \in \bar{\Gamma}$.

The theory thus far applies to any first order predicate logic, pure or applied, provided the primitive symbols are enumerable. We now consider an infinite sequence of applied predicate logics of first order, S_0, S_1, S_2, ... , which have as primitive symbols all the primitive symbols of the system S and in addition certain individual constants. Viz., the primitive symbols of S_o are those of S and the individual constants $W_{0,0}$ $W_{1,0}$ $W_{2,0}$, ... ; the primitive symbols of S_{n+1} are those of S_n and the additional individual constants $W_{0,\ n+1}$, $W_{1,n+1}$, $W_{2,n+1}$, ... Also let S_{ω} be the applied predicate logic which has as its primitive symbols the primitive symbols of all the systems S_0, S_1, S_2, ... All the wffs of S_{ω} can be enumerated and so can the wffs of each S_n by deleting from the enumeration of the wffs of S_{ω} the wffs not in S_n.

Let Γ_0 be a given consistent class of wffs of S_0 which have no free individual (ie. subject) variables. We define the classes Γ_n^m (m and n being positive integers) as follows: Γ_1^0, is $\overline{\Gamma}_0$. If the (m +1) -st wff of S_n, n > 0, has the form (Sx) A(x) and is a member of Γ_n^m, then Γ_n^{m+1} is the class whose members are $A(W_{m,n})$ and the members of Γ_n^m; otherwise Γ_n^{m+1} is Γ_n^m. Also Γ_{n+1}^0 is $\bar{\Delta}_n$ where Δ_n is the union of the classes Γ_n^0, Γ_n^1, Γ_n^2, ..., The members of Γ_n^m are wffs of S_n and Γ_{n+1}^0 is a maximal consistent class of wffs of S_n.

Assume that, for a particular m, Γ_n^m is consistent but Γ_n^{m+1} is inconsistent. Then Γ_n^{m+1} is not the same as Γ_n^m but has the additional member $A(W_{m,n})$. By the inconsistency of Γ_n^{m+1} and the Deduction Theorem, $\Gamma_n^m \vdash A\ (W_{m,n}) \supset B$ and $\Gamma_n^m \vdash$ $A\ (W_{m,n}) \supset \sim B$. Hence, by S, $\Gamma_n^m \vdash \sim T\ A(W_{m,n})$.

Let x be an individual (or subject) variable that does not occur in this proof from hypotheses, and in it replace the constant $W_{n,n}$ everywhere by x. Since $W_{n,n}$ does not occur in any of the members of Γ_n^n, we thus have: $\Gamma_n^n \vdash \sim TA(x)$. By making one or more changes in bound variable, by D.R.I, we have: $\Gamma_n^n \vdash (Ay) \sim TA(y)$, where y is not in A(x). Since in P, $(Ay) \sim A(y) \supset \sim (Sy)\, A(y)$, by R.6, $(Ay) \sim TA(y) \supset \sim (Sy)\, TA(y)$ holds in S. Hence $\Gamma_n^n \vdash \sim (Sy)\, TA(y)$. $(Sx)\, A(x)$ is a member of Γ_n^0 and hence Γ_n^n. Therefore $\Gamma_n^n \vdash (Sy)\, A(y)$. Since, in S, $\vdash A \Rightarrow \vdash TA.$, $\Gamma_n^m \vdash T\,(Sy)\, A(y)$. By T.4, $\Gamma_n^n \vdash (Sy)\, TA(y)$. Hence $\Gamma_n^n \vdash \sim (Sy)\, TA(y)$ and $\Gamma_n^n \vdash (Sy)\, TA(y)$.

Since Γ_n^n is consistent, so is Γ_n^{n+1}. By induction it follows that if Γ_n^0 is consistent then Γ_{n+1}^0 is consistent. By another induction, Γ_n^0 is consistent for all n.

Let Γ_ω be the union of the classes $\Gamma_1^0, \Gamma_2^0, \Gamma_3^0, \ldots$, Then Γ_ω is a maximal consistent class of wffs of S_ω. (For Γ_ω could be inconsistent only if, for some n, Γ_n^0 were inconsistent. Further, if C is a wff of S_ω consistent with Γ_ω, then, for some n, C is a wff of S_n and is consistent with Γ_{n+1}^0; since Γ_{n+1}^0 is a maximal consistent class of wffs of S_n, it follows that, C is a member of Γ_{n+1}^0 and therefore is a member of Γ_ω.)

The following are properties of Γ_ω:

(a) If $A \in \Gamma_\omega$ then $\sim A \notin \Gamma_\omega$.

If $\sim A \in \Gamma_\omega$ then Γ_ω would be inconsistent.

(b) If $SA \in \Gamma_\omega$ and $A \notin \Gamma_\omega$ then $\sim A \in \Gamma_\omega$.

If $A \notin \Gamma_\omega$ then A is inconsistent with Γ_ω and $\Gamma_\omega, A \vdash B$ and $\Gamma_\omega, A \vdash \sim B$.

Hence $\Gamma_\omega \vdash A \supset B$, $\Gamma_\omega \vdash A \supset \sim B$ and $\Gamma_\omega \vdash \sim TA$. Since $\Gamma_\omega \vdash SA$, then $\Gamma_\omega \vdash FA$, i.e. $\Gamma_\omega \vdash T\sim A$. Since, in S, $\vdash TA \Rightarrow \vdash A$, $\Gamma_\omega \vdash \sim A$ and $\sim A \in \Gamma_\omega$.

(c) If $A \notin \Gamma_\omega$ then $\sim TA \in \Gamma_\omega$. This is proved in the proof of (b).

(d) At least one of TA, FA, ~SA is a member of Γ_ω.

Let $TA \notin \Gamma_\omega$, $FA \notin \Gamma_\omega$ and $\sim SA \notin \Gamma_\omega$. Since $STA \in \Gamma_\omega$, $SFA \in \Gamma_\omega$ and $S\sim SA \in \Gamma_\omega$, $\sim TA \in \Gamma_\omega$, $\sim FA \in \Gamma_\omega$ and $\sim\sim SA \in \Gamma_\omega$. By S, $SA \in \Gamma_\omega$ and hence $\Gamma_\omega \vdash \sim TA \,\&\, \sim FA \,\&\, SA$. By S, $\Gamma_\omega \vdash \sim(TA \vee FA \vee \sim SA)$ and also $\Gamma_\omega \vdash TA \vee FA \vee \sim SA$. By the consistency of Γ_ω, the property (d) holds.

(e) At most one of TA, FA, ~SA is a member of Γ_ω.

By S, $TA \supset \sim FA$, $TA \supset SA$, $FA \supset \sim TA$, $FA \supset SA$, $\sim SA \supset \sim TA$, $\sim SA \supset \sim FA$. Hence if one of TA, FA or ~SA is a member of Γ_ω, then the negations of the other two are also members of Γ_ω. By (a), the other two are non-members of Γ_ω.

Now we can make an assignment of values to each member of S . If $TA \in \Gamma_\omega$, then A has T, if $FA \in \Gamma_\omega$ then A has F and if $\sim SA \in \Gamma_\omega$ then A has ~S. If $TG(a_1,\ldots,a_n) \in \Gamma_\omega$ for all choices of $a_1,\ldots,a_n$, then $G(x_1,\ldots,x_n)$ has T. If $SG(a_1,\ldots,a_n) \in \Gamma_\omega$ for all choices of $a_1,\ldots,a_n$, then $G(x_1,\ldots,x_n)$ has S. $G(x_1,\ldots,x_n)$ has F if G has S and has not T. $G(x_1,\ldots,x_n)$ has ~S if G has not S.

To show that the assignment is a consistent one, one must show that the primitive connectives satisfy the matrices and that the quantifiers satisfy their appropriate properties.

(i) ~.

In S, $\Gamma_\omega \vdash TA \Rightarrow \Gamma_\omega \vdash F\sim A$, $\Gamma_\omega \vdash FA \Rightarrow \Gamma_\omega \vdash T\sim A$, $\Gamma_\omega \vdash \sim SA \Rightarrow \Gamma_\omega \vdash \sim S\sim A$.

Hence, if A has T, ~A has F, if A has F, ~A has T, and if A has ~S then ~A has ~S. Hence the connective ~ satisfies the matrix.

(ii) $\supset$.

In S, $\Gamma_\omega \vdash TA$, $\Gamma_\omega \vdash TB \Rightarrow \Gamma_\omega \vdash T(A \supset B)$, $\Gamma_\omega \vdash TA$, $\Gamma_\omega \vdash FB \Rightarrow \Gamma_\omega \vdash F(A \supset B)$, $\Gamma_\omega \vdash TA$, $\Gamma_\omega \vdash {\sim}SB \Rightarrow \Gamma_\omega \vdash {\sim}S(A \supset B)$, $\Gamma_\omega \vdash FA$ or $\Gamma_\omega \vdash {\sim}SA \Rightarrow \Gamma_\omega \vdash T(A \supset B)$.

Hence, if A has T and B has T then A $\supset$ B has T, if A has T and B has F then A $\supset$ B has F, if A has T and B has ~S then A $\supset$ B has ~S, if A has F then A $\supset$ B has T, and if A has ~S then A $\supset$ B has T. Hence the connective $\supset$ satisfies the matrix.

(iii) T_n.

In S, $\Gamma_\omega \vdash TA \Rightarrow \Gamma_\omega \vdash TT_nA$, $\Gamma_\omega \vdash FA \Rightarrow \Gamma_\omega \vdash {\sim}ST_nA$, $\Gamma_\omega \vdash {\sim}SA \Rightarrow \Gamma_\omega \vdash {\sim}ST_nA$.

Hence, if A has T, T_nA has T, if A has F or ~S then T_nA has ~S. Hence the connective T_n satisfies the matrix.

(iv) A.

(a) Let $\Gamma_\omega \vdash TB(w_{m,n})$, for all $w_{m,n}$. Let $T(Ax)B(x) \notin \Gamma_\omega$. Then ${\sim}T(Ax)B(x) \in \Gamma_\omega$ and $\Gamma_\omega \vdash {\sim}T(Ax)B(x)$. By T.2, $(Ax)TB(x) \supset T(Ax)B(x)$ and hence, by P and R.6, ${\sim}T(Ax)B(x) \supset {\sim}(Ax)TB(x)$ and $\Gamma_\omega \vdash {\sim}(Ax)TB(x)$. Since, in P, ${\sim}(Ax)\ A(x) \supset (Sx){\sim}A(x)$, by R.6, ${\sim}(Ax)TB(x) \supset (Sx){\sim}TB(x)$. Hence, $\Gamma_\omega \vdash (Sx){\sim}TB(x)$. By the construction of Γ_ω, $\Gamma_\omega \vdash {\sim}TB(w_{m,n})$ for some $w_{m,n}$. Due to the consistency of Γ_ω, this contradicts the initial assumption and hence, if $B(w_{m,n})$ has T for all $w_{m,n}$ then $(Ax)B(x)$ has T.

(b) Let $\Gamma_\omega \vdash {\sim}SB(w_{m,n})$ for some $w_{m,n}$. Since, in S, $A \supset (Sx)A$,

$\Gamma_\omega \vdash (Sx){\sim}SB(x)$. Since, in P, $(Sx){\sim}A(x) \supset {\sim}(Ax)A(x)$, by R.6, $(Sx){\sim}SB(x) \supset {\sim}(Ax)SB(x)$. Hence $\Gamma_\omega \vdash {\sim}(Ax)SB(x)$. By T.6, $S(Ax)B(x) \supset (Ax)SB(x)$, and hence $\Gamma_\omega \vdash {\sim}S(Ax)B(x)$. Therefore, if $B(x)$ has ~S for some x, then $(Ax)B(x)$ has ~S.

(c) Let $\Gamma_\omega \vdash FB(w_{m,n})$ for some $w_{m,n}$ and $\Gamma_\omega \vdash SB(w_{m,n})$ for all $w_{m,n}$. Hence $\Gamma_\omega \vdash (Sx)FB(x)$. Let $(Ax)SB(x) \notin \Gamma_\omega$. Since, in P, $S(Ax)A(x)$, by R.6, $\Gamma_\omega \vdash S(Ax)SB(x)$. Hence ${\sim}(Ax)SB(x) \in \Gamma_\omega$. Using P and R.6, $(Sx){\sim}SB(x) \in \Gamma_\omega$. Hence, ${\sim}SB(w_{m,n}) \in \Gamma_\omega$ for some $w_{m,n}$. By the consistency of Γ_ω, $(Ax)SB(x) \in \Gamma_\omega$. By T.3, $\Gamma_\omega \vdash F(Ax)B(x)$. Therefore, if $B(x)$ has F for some x and S for all x then $(Ax)B(x)$ has F.

(v) <u>S.</u>

(a) Let $\Gamma_\omega \vdash TB(w_{m,n})$ for some $w_{m,n}$. Then $\Gamma_\omega \vdash (Sx)TB(x)$ and, since, by A.10, $(Sx)TA(x) \supset T(Sx)A(x)$, $\Gamma_\omega \vdash T(Sx)B(x)$. If $B(x)$ has T for some x, then $(Sx)B(x)$ has T.

(b) Let $\Gamma_\omega \vdash {\sim}SB(w_{m,n})$ for all $w_{m,n}$. Let $(Ax){\sim}SB(x) \notin \Gamma_\omega$. By P and R.6, $S(Ax){\sim}SB(x)$ and hence ${\sim}(Ax){\sim}SB(x) \in \Gamma_\omega$. By P and R.6, ${\sim}(Ax){\sim}SB(x) \supset (Sx)SB(x)$ and hence $(Sx)SB(x) \in \Gamma_\omega$. By the construction of Γ_ω, $SB(w_{m,n}) \in \Gamma_\omega$ for some $w_{m,n}$. By the consistency of Γ_ω, $(Ax){\sim}SB(x) \in \Gamma_\omega$. By P and R.6, ${\sim}(Sx)SB(x) \in \Gamma_\omega$. By T.5, ${\sim}S(Sx)B(x) \in \Gamma_\omega$. If $B(x)$ has ~S for all x then $(Sx)B(x)$ has ~S.

(c) Let $\Gamma_\omega \vdash FB(w_{m,n})$ for some $w_{m,n}$ and let $TB(w_{m,n}) \notin \Gamma_\omega$ for all $w_{m,n}$. Then $\Gamma_\omega \vdash (Sx)FB(x)$. Let $(Ax){\sim}TB(x) \notin \Gamma_\omega$. Then, since $S(Ax){\sim}TB(x)$, ${\sim}(Ax){\sim}TB(x) \in \Gamma_\omega$. Since ${\sim}(Ax){\sim}TB(x) \supset (Sx)TB(x)$, $(Sx)TB(x) \in \Gamma_\omega$. Hence $TB(w_{m,n}) \in \Gamma_\omega$ for some $w_{m,n}$. This is a contradiction. Hence $(Ax){\sim}TB(x)$

$\in \Gamma_{\omega}$. By A.11, $\Gamma_{\omega} \vdash F(Sx)B(x)$. If $B(x)$ has F for some x and has not T for all x then $(Sx)B(x)$ has F.

Since Γ_0 was chosen as an arbitrary consistent class of wffs of S_0 without free individual variables, <u>every consistent class of well-formed formulae of S_0 without free individual variables is simultaneously satisfiable in a denumerable domain.</u>

Let B be a valid wff. Then the class consisting of ~B only is not simultaneously satisfiable and hence is not consistent. Therefore, for some wff A, $\sim B \vdash A$ and $\sim B \vdash \sim A$. By the Deduction Theorem, $\vdash B$.

This completeness proof assumed the consistency of the system S. This can be easily shown by considering the domain of one individual. Then there is no difference between $(Ax)A(x)$, $(Sx)A(x)$ and $A(x_0)$, where x_0 is that individual. Thus all the quantifiers can be removed and the system reduced to a sentential one. This sentential system is consistent because the axioms are valid according to the matrices and the rules preserve this validity.

<u>Meta-theorem 3.</u>

Substitutivity of Equivalents holds in S for $\simeq$. That is, if $\vdash_S A \simeq B$ then $\vdash_S C(A) \simeq C(B)$, where substitution into C can be made for any argument place.

<u>Proof.</u> The proof is an extension of the one given for Meta-theorem 3 of Chapter 1. If $\vdash_S A \simeq B$ then $\vdash_S (Ax)A \simeq (Ax)B$ and $\vdash_S (Sx)A \simeq (Sx)B$,

using Meta-theorem 2.

Meta-theorem 4.

Let B be a wff of S containing only the connectives, $\supset$, T_n, T, &, v, ~T and the quantifiers A, S. Let A be a wff of P obtained from B by deleting any T's or T_n's and replacing ~T by ~. Then, if $\vdash_P$ A then $\vdash_S$ B.

Proof. The proof is the same as that of Meta-theorem 2 except that, instead of there being three values T, F and ~S, there are two values T and ~T. Because of the completeness of the axiomatisation all the necessary theorems of S are available.

Meta-theorem 5.

Let C be a wff of S containing only the connectives, $\supset$, T_n, T, &, v, ~T and the quantifiers A, S, as in Meta-theorem 4. Then, if $\vdash_S A \equiv B$ then $\vdash_S C(A) \equiv C(B)$, where substitution into C can be made for any argument place.

Proof. The proof is an extension of the one given for Meta-theorem 5 of Chapter 1. If $\vdash_S A \equiv B \Rightarrow \vdash_S (Ax)A \equiv (Ax)B$ and $\vdash_S (Sx)A \equiv (Sx)B$, by Meta-theorem 2.

Meta-theorem 6.

If the domain is restricted to all x's such that D(x) is true, then the axioms and rules restricted to this domain will still hold, provided that the domain is non-empty, i.e. $\vdash_S (Sx)D(x)$.

Proof. The sentential axioms and rules still hold, by repeated use of $\vdash_S A \supset . B \supset A$.

(a) Axiom 9 becomes : $(Sx)(T_nD(x) \& \sim SA(x)) \supset \sim S(Ax)(D(x) \supset A(x))$, with variables restricted to $D(x)$.

Let $(Sx)(T_nD(x) \& \sim SA(x))$ take the value 1. For some x_0, $T_nD(x_0) \& \sim SA(x_0)$ takes the value 1. Hence $D(x_0)$ takes the value 1 and $A(x_0)$ takes the value n. Hence $D(x_0) \supset A(x_0)$ has the value n and $(Ax)(D(x) \supset A(x))$ also has the value n. Therefore $\sim S(Ax)(D(x) \supset A(x))$ takes the value 1 and the above wff is valid and hence provable in S.

(b) Axiom 10 becomes : $(Sx)(T_nD(x) \& TA(x)) \supset T(Sx)(T_nD(x) \& A(x))$.
Let $(Sx)(T_nD(x) \& TA(x))$ take the value 1. Then $D(x_0)$ and $A(x_0)$ both have the value 1, for some x_0. Hence $T_nD(x_0) \& A(x_0)$ has the value 1 and so has $(Sx)(T_nD(x) \& A(x))$ and $T(Sx)(T_nD(x) \& A(x))$. Therefore the above wff is valid and hence provable in S.
(c) Axiom 11 becomes : $(Ax)(D(x) \supset \sim TA(x)) \& (Sx)(T_nD(x) \& FA(x)) \supset F(Sx)(T_nD(x) \& A(x))$.

Let $(Ax)(D(x) \supset \sim TA(x))$ and $(Sx)(T_nD(x) \& FA(x))$ have the value 1. Then $D(x_0)$ has the value 1 and $A(x_0)$ has the value 0, for some x_0. Also, for all x, if $D(x)$ has the value 1 then $A(x)$ has the value 0 or n. Hence, for some x_0, $T_nD(x_0) \& A(x_0)$ has the value 0. It is not the case that $T_nD(x)$ and $A(x)$ are both true for some x. Hence $(Sx)(T_nD(x) \& A(x))$ is false and the above wff is valid and hence provable in S.

(d) Axiom 12 becomes : $(Ax)(D(x) \supset \sim SA(x)) \supset \sim S(Sx)(T_nD(x) \& A(x))$.
Let $(Ax)(D(x) \supset \sim SA(x))$ take the value 1. Then, for all x, if

$D(x)$ has the value 1 then $A(x)$ has the value n. Hence $T_nD(x)$ & $A(x)$ has the value n, for all x. Hence $(Sx)(T_nD(x) \& A(x))$ has the value n and therefore the above wff is valid and hence provable in S.

(e) Axiom 13 becomes : $D(x) \supset . (Ax)(D(x) \supset A(x)) \supset A(x)$.
Let $D(x_0)$ take the value 1 and let $(Ax)(D(x) \supset A(x))$ also take the value 1. Then, for all x, if $D(x)$ takes the value 1 then $A(x)$ takes the value 1. Hence $A(x_0)$ takes the value 1. Since this holds for all x_0, the above wff is valid and hence provable in S.

(f) Axiom 14 becomes : $D(x) \supset . A(x) \supset (Sx)(T_nD(x) \& A(x))$.
Let $D(x_0)$ and $A(x_0)$ take the value 1. Then $T_nD(x_0) \& A(x_0)$ takes the value 1 and so does $(Sx)(T_nD(x) \& A(x))$. Hence the above wff is valid and provable in S.

(g) The Rule 4 works just as well with restricted variables.

(h) Rule 5 becomes : $\vdash_S D(x) \supset . D(z_1) \supset . D(z_2) \supset . \ldots . \supset D(z_n) \supset . A \supset B(x) \Rightarrow \vdash_S D(z_1) \supset . D(z_2) \supset . \ldots . \supset D(z_n) \supset . A \supset (Ax)(D(x) \supset B(x))$, where $z_1, \ldots, z_n$ are all the free variables in $A \supset B$ besides the variable x.

Let $D(z_1)$, $D(z_2)$, ..., $D(z_n)$ and A all take the value 1. If $D(x_0)$, $D(z_1)$, $D(z_2)$, ..., $D(z_n)$ and A all take the value 1 then $B(x_0)$ will take the value 1. Since this holds for all values x, $(Ax)(D(x) \supset B(x))$ takes the value 1 and hence Rule 5 still holds.

(i) Rule 6 becomes : $\vdash_P D(z_1) \supset . \ldots . \supset D(z_n) \supset . \mathcal{A}'(A_1(x_{1,1}, \ldots,$

$x_{1,i_1}), \ldots, A_n(x_{n,1}, \ldots, x_{n,i_n}))$, where $z_1, \ldots, z_n$ are all the free variables in $\mathcal{A}(A_1, \ldots, A_n)$, $A_1, \ldots, A_n$ are the only wff-schemata in $\mathcal{A}$, $x_{j,1}, \ldots, x_{j,i_j}$ are the only variables in A_j, and $\mathcal{A}'$ contains all the quantifier restrictions on the bound variables of the A_j's, $\Rightarrow \vdash_S D(z_1) \supset. \ldots. \supset D(z_n) \supset. (Ax_{1,1})(D(x_{1,1}) \supset. (Ax_{1,2})(D(x_{1,2}) \supset. \ldots. \supset (x_{1,i_1})(D(x_{1,i_1}) \supset. S\mathcal{B}_1'(x_{1,1}, \ldots, x_{1,i_1})) \ldots.) \supset. \ldots. \supset (Ax_{n,1})(D(x_{n,1}) \supset. (Ax_{n,2})(D(x_{n,2}) \supset. \ldots. \supset (Ax_{n,i_n})(D(x_{n,i_n}) \supset. S\mathcal{B}_n'(x_{n,1}, \ldots, x_{n,i_n})) \ldots.) \supset \mathcal{A}'(\mathcal{B}_1'(x_{1,1}, \ldots, x_{1,i_1}), \ldots, \mathcal{B}_n'(x_{n,1}, \ldots, x_{n,i_n}))$, where $x_{j,1}, \ldots, x_{j,i_j}$ are the only free variables in $\mathcal{B}_j'$ and $\mathcal{B}_j'$ contains the quantifier restrictions on its bound variables.

Let $D(z_1), \ldots, D(z_n)$, $(Ax_{1,1}) \supset. \ldots. \supset (Ax_{1,i_1})(D(x_{1,i_1}) \supset S\mathcal{B}_1'(x_{1,1}, \ldots, x_{1,i_1})) \ldots.), \ldots, (Ax_{n,1})(D(x_{n,1}) \supset. \ldots. \supset (Ax_{n,i_n})(D(x_{n,i_n}) \supset S\mathcal{B}_n'(x_{n,1}, \ldots, x_{n,i_n})) \ldots.)$, all have the value 1.

So, if $D(x_{j,k})$ has the value 1 for all variables $x_{j,k}$, then $S\mathcal{B}_j'$ has the value 1 and $\mathcal{B}_j'$ has the value 1 or 0. Let us now examine the restricted quantification in $\mathcal{A}'$, given that the $\mathcal{B}_j'$'s have the value 1 or 0.

(a) $(Ax)(D(x) \supset C(x))$, where $C(x)$ is 2-valued, will be 2-valued and its value will be independent of whether one lets $D(x)$ take the value n or not.

(b) $(Sx)(T_n D(x) \& C(x))$, where $C(x)$ is 2-valued, will be 2-valued and its value will be independent of whether one lets $D(x)$ take the value n or not and independent of whether T_n is deleted or not.

Similarly with the connectives of $\mathcal{A}'$, it does not affect the final value if they are regarded as 2-valued instead of 3-valued. So, since $\mathcal{A}'(A_1,\ldots,A_n)$ has the value 1 in P, $\mathcal{A}'(\mathcal{B}_1',\ldots,\mathcal{B}_n')$ has the value 1 in S.

Meta-theorem 7.

If A is a wff of S containing only the connectives ~, &, v, T and the quantifiers A, S, then there is a wff A' of S such that $A \simeq A'$ and A' has all of its quantifiers, A, S, ∀, E at the beginning of the formula.

Proof. The following are valid and hence provable in S :

$\sim(Ax)A \simeq (Ex)\sim A$.

$\sim(Sx)A \simeq (\forall x)\sim A$.

$\sim(\forall x)A \simeq (Sx)\sim A$.

$\sim(Ex)A \simeq (Ax)\sim A$.

$(Ax)A \& B \simeq (Ax)(A \& B)$, where x is not free in B.

$(Sx)A \& B \simeq (Sx)(A \& B)$, where x is not free in B.

$(\forall x)A \& B \simeq (\forall x)(A \& B)$, where x is not free in B.

$(Ex)A \& B \simeq (Ex)(A \& B)$, where x is not free in B.

$(Ax)A \vee B \simeq (Ax)(A \vee B)$, where x is not free in B.

$(Sx)A \vee B \simeq (Sx)(A \vee B)$, where x is not free in B.

$(\forall x)A(x) \vee B \simeq (\forall x)(Sy)(Sz)(Aw){\sim}({\sim}(A(x) \vee B) \;\&\; {\sim}(T{\sim}B \;\&\; {\sim}SA(y) \;\&$
$TA(z) \;\&\; {\sim}T{\sim}A(w)))$, where x is not free in B.

$(Ex)A(x) \vee B \simeq (Ex)(Ay)(Az)((A(x) \vee B) \;\&\; {\sim}(T{\sim}B \;\&\; {\sim}SA(y) \;\&\; TA(z)))$,
where x is not free in B.

$T(Ax)A \simeq (Ax)TA$.

$T(Sx)A \simeq (Sx)TA$.

$T(\forall x)A(x) \simeq (Sx)(Ay)(TA(x) \;\&\; {\sim}T{\sim}A(y))$.

$T(Ex)A(x) \simeq (Sx)(Ay)(TA(x) \;\&\; SA(y))$.

Applying these equivalences to each connective in turn, one can construct an $A' \simeq A$ such that the quantifiers are in front of a formula containing connectives only.

Again, Rosser and Turquette's axiomatisation in [21] could have been used but I give an axiomatisation using Rule 6, due to Routley.

(ii) <u>The 3-valued Lukasiewicz Logic.</u>

The quantifiers A and S that I will use are the ones of the 3-valued Lukasiewicz predicate logic, satisfying the following properties :

If A(x) is true for all x then (Ax)A(x) is true; if A(x) is false for some x then (Ax)A(x) is false; and if A(x) is not false, for all x, and paradoxical for some x then (Ax)A(x) is paradoxical.
If A(x) is true for some x then (Sx)A(x) is true; if A(x) is false

for all x then (Sx)A(x) is false; and if A(x) is not true, for all x, and paradoxical for some x then (Sx)A(x) is paradoxical.
The quantifiers A and S are interdefinable, (Sx)A(x) being definable as ~(Ax)~A(x).

The formal axiomatic system L is obtained by adding the following to the sentential system L :

Primitives.

3. x,y,z,..... (subject variables)

4. f,g,h,..... (predicate variables)

5. A (universal quantifier)

Formation Rules.

3. If f is an n-ary predicate variable and $x_1,\ldots,x_n$ are subject variables then $f(x_1,\ldots,x_n)$ is a wff.

4. If A is a wff and x is a subject variable then (Ax)A is a wff.

Definition.

$(Sx)A =_{df} \sim(Ax)\sim A.$

Axioms.

[All the sentential axioms are written in schematic form.]

5. $(Ax)A \rightarrow A.$

6. $(Ax)TA(x) \rightarrow T(Ax)A(x).$

7. $F(Ax)A(x) \rightarrow (Sx)FA(x).$

Rules.

3. Substitution for subject variables, free and bound, with the provisos in [2], p192.

4. $\vdash_L A \rightarrow B \Rightarrow \vdash_L A \rightarrow (Ax)B$, where x is not free in A.

Theorems.

D.R.1. $\vdash_L A \Rightarrow \vdash_L (Ax)A$.

R.4 : $\vdash_L (p \supset p) \rightarrow A \Rightarrow \vdash_L (p \supset p) \rightarrow (Ax)A$ ______(1)

(1), L : $(p \supset p) \rightarrow (Ax)A$ ______(2)

(2), L : $(Ax)A$.

D.R.2. $\vdash_L A \rightarrow B \Rightarrow \vdash_L (Sx)A \rightarrow B$, if x is not free in B.

L : $\sim B \rightarrow \sim A$ ______(1)

(1), R.4 : $\sim B \rightarrow (Ax)\sim A$ ______(2)

(2), L : $\sim(Ax)\sim A \rightarrow B$ ______(3)

(3), Defn. S : $(Sx)A \rightarrow B$.

1. $(Sx)A(x) \leftrightarrow \sim(Ax)\sim A(x)$.

L : $\sim(Ax)\sim A(x) \leftrightarrow \sim(Ax)\sim A(x)$ ______(1)

(1), Defn. S : $(Sx)A(x) \leftrightarrow \sim(Ax)\sim A(x)$.

2. $\sim(Sx)A(x) \leftrightarrow (Ax)\sim A(x)$.

T.1, L : $\sim(Sx)A(x) \leftrightarrow (Ax)\sim A(x)$.

3. $(Sx)\sim A(x) \leftrightarrow \sim(Ax)A(x)$.

T.1, L : $(Sx)\sim A(x) \leftrightarrow \sim(Ax)\sim\sim A(x)$ ______(1)

(1), L : $(Sx)\sim A(x) \leftrightarrow \sim(Ax)A(x)$.

4. $\sim(Sx)\sim A(x) \leftrightarrow (Ax)A(x)$.

T.3, L : $\sim(Sx)\sim A(x) \leftrightarrow (Ax)A(x)$.

5. $A \rightarrow (Sx)A$.

A.5, L : $\sim A \rightarrow \sim(Ax)A$ ______(1)

(1), L : $\sim\sim A \rightarrow \sim(Ax)\sim A$ ______(2)

(2), L, Defn. S : $A \rightarrow (Sx)A$.

6. $\underline{(Ax)A \rightarrow (Sx)A.}$

A.5, T.5, L : $(Ax)A \rightarrow (Sx)A$.

7. $\underline{T(Ax)A(x) \rightarrow (Ax)TA(x).}$

A.5, L : $T(Ax)A \rightarrow TA$ _______ (1)

(1), R.4 : $T(Ax)A \rightarrow (Ax)TA$.

8. $\underline{(Sx)FA(x) \rightarrow F(Ax)A(x).}$

A.5, L : $\sim A(x) \rightarrow \sim(Ax)A(x)$ _______ (1)

(1), L : $T\sim A(x) \rightarrow T\sim(Ax)A(x)$ _______ (2)

(2), Defn. F, D.R.2 : $(Sx)FA(x) \rightarrow F(Ax)A(x)$.

D.R.3. $\underline{\vdash_L A \rightarrow B \Rightarrow \vdash_L (Sx)A \rightarrow (Sx)B.}$

T.5, L : $A \rightarrow (Sx)B$ _______ (1)

(1), D.R.2 : $(Sx)A \rightarrow (Sx)B$.

D.R.4. $\underline{\vdash_L A \rightarrow B \Rightarrow \vdash_L (Ax)A \rightarrow (Ax)B.}$

A.5, L : $(Ax)A \rightarrow B$ _______ (1)

(1), R.4 : $(Ax)A \rightarrow (Ax)B$.

9. $\underline{(Sx)(A \vee B) \leftrightarrow (Sx)A \vee (Sx)B.}$

L : $A \rightarrow A \vee B$ _______ (1)

(1), D.R.3 : $(Sx)A \rightarrow (Sx)(A \vee B)$ _______ (2)

L, D.R.3 : $(Sx)B \rightarrow (Sx)(A \vee B)$ _______ (3)

(2), (3), L : $(Sx)A \vee (Sx)B \rightarrow (Sx)(A \vee B)$ _______ (4)

A.5 : $A \rightarrow (Sx)A$ _______ (5)

A.5 : $B \rightarrow (Sx)B$ _______ (6)

(5), (6), L : $A \vee B \rightarrow (Sx)A \vee (Sx)B$ _______ (7)

(7), D.R.2 : $(Sx)(A \vee B) \rightarrow (Sx)A \vee (Sx)B$ _______ (8)

(4), (8), L : (Sx)(A v B) ⟷ (Sx)A v (Sx)B.

10. <u>P(Ax)A(x) ⟷ (Ax)~FA(x) & (Sx)PA(x).</u>

L : P(Ax)A(x) ⟷ ~T(Ax)A(x) & ~F(Ax)A(x) ______(1)

A.6, T.7, L : ~T(Ax)A(x) ⟷ ~(Ax)TA(x) ______(2)

(2), T.3, L : ~T(Ax)A(x) ⟷ (Sx)~TA(x) ______(3)

A.7, T.8, L : ~F(Ax)A(x) ⟷ ~(Sx)FA(x) ______(4)

(4), T.2, L : ~F(Ax)A(x) ⟷ (Ax)~FA(x) ______(5)

(1), (3), (5), L : P(Ax)A(x) ⟷ (Sx)~TA(x) & (Ax)~FA(x) ______(6)

L, D.R.3 : (Sx)~TA(x) ⟷ (Sx)(FA(x) v PA(x)) ______(7)

(7), T.9, L : (Sx)~TA(x) ⟷ (Sx)FA(x) v (Sx)PA(x) ______(8)

(6), (8), L : P(Ax)A(x) ⟷ ((Sx)FA(x) & (Ax)~FA(x)) v ((Sx)PA(x) & (Ax)~FA(x)) ______(9)

A.2, A.7, T.8, L : (Sx)FA(x) & (Ax)~FA(x) ⟷ F(Ax)A(x) & ~F(Ax)A(x) ______(10)

(9), (10), L : P(Ax)A(x) ⟷ (Ax)~FA(x) & (Sx)PA(x).

11. <u>(Ax)CA(x) → C(Ax)A(x).</u>

T.10, L : P(Ax)A(x) → (Sx)PA(x) ______(1)

(1), L : ~(Sx)PA(x) → ~P(Ax)A(x) ______(2)

L : PA(x) ⟷ ~CA(x) ______(3)

(3), D.R.3 : (Sx)PA(x) ⟷ (Sx)~CA(x) ______(4)

(4), L, Defn.A : ~(Sx)PA(x) ⟷ (Ax)CA(x) ______(5)

(2), (5), L : (Ax)CA(x) → C(Ax)A(x).

12. <u>T(Sx)A(x) ⟷ (Sx)TA(x).</u>

A.7, T.8 : F(Ax)A(x) ⟷ (Sx)FA(x) ______(1)

(1), Defn.F : $T\sim(Ax)A(x) \leftrightarrow (Sx)T\sim A(x)$ ______(2)

T.3, L : $T\sim(Ax)A(x) \leftrightarrow T(Sx)\sim A(x)$ ______(3)

(2), (3), L : $T(Sx)A(x) \leftrightarrow (Sx)TA(x)$.

13. $\underline{F(Sx)A(x) \leftrightarrow (Ax)FA(x).}$

A.6, T.7 : $T(Ax)A(x) \leftrightarrow (Ax)TA(x)$ ______(1)

(1), L, Defn.F : $T(Ax)\sim A(x) \leftrightarrow (Ax)FA(x)$ ______(2)

(2), T.2, L : $T(Ax)\sim A(x) \leftrightarrow T\sim(Sx)A(x)$ ______(3)

(2), (3), Defn.F : $F(Sx)A(x) \leftrightarrow (Ax)FA(x)$.

14. $\underline{P(Sx)A(x) \leftrightarrow (Sx)PA(x)\ \&\ (Ax)\sim TA(x).}$

L : $P(Sx)A(x) \leftrightarrow \sim T(Sx)A(x)\ \&\ \sim F(Sx)A(x)$ ______(1)

T.12, L : $\sim T(Sx)A(x) \leftrightarrow \sim(Sx)TA(x)$ ______(2)

(2), T.2, L : $\sim T(Sx)A(x) \leftrightarrow (Ax)\sim TA(x)$ ______(3)

T.13, L : $\sim F(Sx)A(x) \leftrightarrow \sim(Ax)FA(x)$ ______(4)

(4), T.3, L : $\sim F(Sx)A(x) \leftrightarrow (Sx)\sim FA(x)$ ______(5)

L : $\sim FA(x) \leftrightarrow TA(x)\ v\ PA(x)$ ______(6)

(6), D.R.3 : $(Sx)\sim FA(x) \leftrightarrow (Sx)(TA(x)\ v\ PA(x))$ ______(7)

T.9 : $(Sx)(TA(x)\ v\ PA(x)) \leftrightarrow (Sx)TA(x)\ v\ (Sx)PA(x)$ ______(8)

(5), (7), (8), L : $\sim F(Sx)A(x) \leftrightarrow (Sx)TA(x)\ v\ (Sx)PA(x)$ ______(9)

(1), (3), (9), L : $P(Sx)A(x) \leftrightarrow (Ax)\sim TA(x)\ \&\ ((Sx)TA(x)\ v\ (Sx)PA(x))$

______(10)

(10), L : $P(Sx)A(x) \leftrightarrow ((Ax)\sim TA(x)\ \&\ (Sx)TA(x))\ v\ ((Ax)\sim TA(x)\ \&$
$(Sx)PA(x))$ ______(11)

T.12, T.2, L : $(Ax)\sim TA(x)\ \&\ (Sx)TA(x) \leftrightarrow \sim T(Sx)A(x)\ \&\ T(Sx)A(x)$

______(12)

(11), (12), L : $P(Sx)A(x) \leftrightarrow (Sx)PA(x)$ & $(Ax){\sim}TA(x)$.

15. $(Ax)CA(x) \rightarrow C(Sx)A(x)$.

T.14, L : $P(Sx)A(x) \rightarrow (Sx)PA(x)$ ______(1)

(1), L : ${\sim}(Sx)PA(x) \rightarrow {\sim}P(Sx)A(x)$ ________(2)

L : $PA(x) \leftrightarrow {\sim}CA(x)$ ________(3)

(3), D.R.3 : $(Sx)PA(x) \leftrightarrow (Sx){\sim}CA(x)$ ______(4)

(2), (4), L : ${\sim}(Sx){\sim}CA(x) \rightarrow C(Sx)A(x)$ ________(5)

(5), T.4 : $(Ax)CA(x) \rightarrow C(Sx)A(x)$.

Meta-theorem 8.

The Deduction Theorem holds in L for $\supset$, i.e. if $A_1,\dots,A_n \vdash_L B$ then $A_1,\dots,A_{n-1} \vdash_L A_n \supset B$, provided Rule 4 is not used to generalise on any variable of A_n and Rule 3 is not used to change a free variable of A_n.

Proof. The proof is an extension of one used to prove the Deduction Theorem for the sentential system L. Account must be taken of the Rules 3 and 4. If $\vdash_L A_n \supset C(x)$ then by Rule 3, with the proviso in [2], p192. $\vdash_L A_n \supset C(y)$, where x and y are free variables. If $\vdash_L A_n \supset C((Ax)D(x))$ then by Rule 3, with the proviso in [2], p192. $\vdash_L A_n \supset C((Ay)D(y))$. If $\vdash_L A_n \supset . A \rightarrow B$, then by the sentential L, $\vdash_L TA_n$ & $A \rightarrow B$. Since x is not free in TA_n & A, $\vdash_L TA_n$ & $A \rightarrow (Ax)B$, by Rule 4. By the sentential L, $\vdash_L A_n \supset . A \rightarrow (Ax)B$. Hence the Deduction Theorem follows.

Meta-theorem 9.

If B is valid according to the 3-valued matrices and the properties stated for A and S, then B is a thesis of the axiomatic system L.

Proof. The proof is similar to that of Meta-theorem 2. I will indicate the differences between that proof and this.

In showing that Γ_n^{m+1} is consistent if Γ_n^m is, the following theorems and derived rules of L are required :

$\vdash_L A \supset B,\ \vdash_L A \supset {\sim}B \Rightarrow \vdash_L {\sim}TA.$

$\vdash_L A(x) \Rightarrow \vdash_L (Ax)A(x).$ (D.R.1)

$(Ax){\sim}A \rightarrow {\sim}(Sx)A.$ (T.2)

$\vdash_L A \Rightarrow \vdash_L TA.$

$T(Sx)A(x) \rightarrow (Sx)TA(x).$ (T.12)

The property (b) of Γ_ω is : If $CA \in \Gamma_\omega$ and $A \notin \Gamma_\omega$ then ${\sim}A \in \Gamma_\omega$. To show that this property holds, we need the following :

$\vdash_L CA,\ \vdash_L {\sim}TA \Rightarrow \vdash_L FA.$

$\vdash_L TA \Rightarrow \vdash_L A.$

The property (d) of Γ_ω is : At least one of TA, FA, PA is a member of Γ_ω. To prove this, we need the following : CTA, CFA, CPA, TA v FA v PA, all theorems of L.

The property (e) of Γ_ω is : At most one of TA, FA, PA is a member of Γ_ω. To prove this, we need the following : $TA \supset {\sim}FA$, $TA \supset {\sim}PA$, $FA \supset {\sim}TA$, $FA \supset {\sim}PA$, $PA \supset {\sim}TA$, $PA \supset {\sim}FA$, all theorems of L.

For the assignment of values, if $PA \in \Gamma_\omega$ then A has P. If $FG(a_1, \ldots, a_n) \in \Gamma_\omega$ for some choice of $a_1, \ldots, a_n$, then $G(x_1, \ldots, x_n)$ has F. If G has not T and has not F then G has P.

In checking the connectives, ~ and →, one needs the following :

$TA \supset F{\sim}A$, $FA \supset T{\sim}A$, $PA \supset P{\sim}A$, $TA \,\&\, TB \supset T(A \rightarrow B)$, $TA \,\&\, PB \supset P(A \rightarrow B)$, $TA \,\&\, FB \supset F(A \rightarrow B)$, $PA \,\&\, TB \supset T(A \rightarrow B)$, $PA \,\&\, PB \supset T(A \rightarrow B)$, $PA \,\&\, FB \supset P(A \rightarrow B)$, $FA \supset T(A \rightarrow B)$, all theorems of L.

In checking the quantifier, A, in part (a) one needs the following : $(Ax)TA(x) \rightarrow T(Ax)A(x)$, $\sim(Ax)A \rightarrow (Sx){\sim}A$, both theorems of L (A.6, T.3).

Part (b) becomes : Let $\Gamma_\omega \vdash FB(w_{m,n})$ for some $w_{m,n}$. Hence $\Gamma_\omega \vdash (Sx) FB(x)$, since $A \rightarrow (Sx)A$ holds in L. By T.8, $\Gamma_\omega \vdash F(Ax)B(x)$. Therefore, if $B(x)$ has F for some x, then $(Ax)B(x)$ has F.

Part (c) becomes : Let $\Gamma_\omega \vdash PB(w_{m,n})$ for some $w_{m,n}$, and $\Gamma_\omega \vdash {\sim}FB(w_{m,n})$ for all $w_{m,n}$. Since $A \rightarrow (Sx)A$ holds in L, $\Gamma_\omega \vdash (Sx)PB(x)$. Let $(Ax){\sim}FB(x) \notin \Gamma_\omega$. Since $\vdash_L C{\sim}FB(x)$ and by T.11, $\vdash_L C(Ax){\sim}FB(x)$. Hence $\sim(Ax){\sim}FB(x) \in \Gamma_\omega$. By the definition of S, $(Sx)FB(x) \in \Gamma_\omega$. Hence $FB(w_{m,n}) \in \Gamma_\omega$, contradicting the consistency of Γ_ω. Therefore, $(Ax){\sim}FB(x) \in \Gamma_\omega$ and $\Gamma_\omega \vdash (Ax){\sim}FB(x)$. By T.10, $\Gamma_\omega \vdash P(Ax)B(x)$. Therefore, if $B(x)$ has not F for all x and has P (i.e. has not T) for some x then $(Ax)B(x)$ has P.

The completeness of the axiomatic system L can now be proved as can its consistency, which follows by the same procedure as used in Meta-theorem 2.

Meta-theorem 10.

Substitutivity of Equivalents holds in L for $\leftrightarrow$. That is, if $\vdash_L A \leftrightarrow B$ then $\vdash_L C(A) \leftrightarrow C(B)$, where substitution into C can be made for any argument place.

Proof. Since $\vdash_L A \leftrightarrow B \Rightarrow \vdash_L (Ax)A \leftrightarrow (Ax)B$, by D.R.4, the theorem follows as before.

Meta-theorem 11.

Let B be a wff of L containing only the connectives, $\supset$, T, &, v, ~T, and quantifiers A, S. Let A be a wff of P obtained from B by deleting any T's and replacing ~T by ~. Then, if $\vdash_P A$ then $\vdash_L B$.

Proof. The proof is the same as that of Meta-theorem 9 except that, instead of there being three values, T, F, and P, there are two values, T and ~T. All the necessary theorems are available because of the completeness of the system L.

Meta-theorem 12.

Let C be a wff of S containing only the connectives, $\supset$, T, &, v, ~T and the quantifiers A, S, as in Meta-theorem 11. Then, if $\vdash_L A \equiv B$ then $\vdash_L C(A) \equiv C(B)$, where substitution into C can be made for any argument place.

Proof. Since $\vdash_L A \equiv B \Rightarrow \vdash_L (Ax)A \equiv (Ax)B$ and $\vdash_L (Sx)A \equiv (Sx)B$, because validity is preserved, the theorem follows as before.

Meta-theorem 13.

If the domain is restricted to all x's such that D(x) is true, then the axioms and rules restricted to this domain will still hold, provided that the domain is non-empty, i.e. $\vdash_L (Sx)D(x)$.

Proof. The variables are restricted as follows : (Sx)(TD(x) & A(x)) is the restriction of (Sx)A(x) to D(x). (Ax)(D(x) $\supset$ A(x)) is the restriction of (Ax)A(x) to D(x).

The proof follows the lines of that of Meta-theorem 6.

(iii) The 4-valued Significance Logic.

The quantifiers A and S that I will use are the ones satisfying the following properties :

If A(x) is true for all x then (Ax)A(x) is true, if A(x) is true or paradoxical for all x and A(x) is paradoxical for some x then (Ax)A(x) is paradoxical, if A(x) is significant for all x and false for some x then (Ax)A(x) is false, and if A(x) is non-significant for some x then (Ax)A(x) is non-significant.

If A(x) is true for some x them (Sx)A(x) is true, if A(x) is not true for all x and paradoxical for some x then (Sx)A(x) is paradoxical, if A(x) is false or non-significant for all x and false for some x then (Sx)A(x) is false, and if A(x) is non-significant for all x them (Sx)A(x) is non-significant.

The formal axiomatic system LS is obtained by adding the following to the sentential system LS :

Primitives.

3. x,y,z,.....(subject variables)

4. f,g,h,.....(predicate variables)

5. A (universal quantifier), S (existential quantifier).

Formation Rules.

3. If f is an n-ary predicate variable and $x_1,\dots,x_n$ are subject variables then $f(x_1,\dots,x_n)$ is a wff.

4. If A is a wff and x is a subject variable then (Ax)A and (Sx)A are wffs.

Definitions.

$(Ex)A =_{df} \sim(Ax)\sim A$.

$(\forall x)A =_{df} \sim(Sx)\sim A$.

Axioms.

[All the sentential axioms are written in schematic form.]

12. $(Ax)A(x) \supset A(x)$.

13. $A(x) \supset (Sx)A(x)$.

14. $(Sx)\sim SA(x) \supset \sim S(Ax)A(x)$.

15. $(Sx)TA(x) \supset T(Sx)A(x)$.

16. $(Ax)\sim TA(x)\ \&\ (Sx)PA(x) \supset P(Sx)A(x)$.

17. $(Ax)(FA(x) \vee \sim SA(x))\ \&\ (Sx)FA(x) \supset F(Sx)A(x)$.

18. $(Ax)\sim SA(x) \supset \sim S(Sx)A(x)$.

Rules.

4. Substitution for subject variables, free and bound, *with the provisos in [23], p192.*

5. $\vdash_{LS} A \supset B \Rightarrow \vdash_{LS} A \supset (Ax)B$, where x is not free in A.

6. $\vdash_{L} \mathcal{A}(A_1(x_{1,1},\ldots,x_{1,i_1}),\ldots\ldots, A_n(x_{n,1},\ldots,x_{n,i_n})) \Rightarrow \vdash_{LS} (Ax_{1,1},\ldots,x_{1,i_1})S\mathcal{B}_1(x_{1,1},\ldots,x_{1,i_1}) \supset. \ldots\ldots \supset (Ax_{n,1},\ldots,x_{n,i_n})S\mathcal{B}_n(x_{n,1},\ldots,x_{n,i_n}) \supset \mathcal{A}(\mathcal{B}_1(x_{1,1},\ldots,x_{1,i_1}),\ldots\ldots,\mathcal{B}_n(x_{n,1},\ldots,x_{n,i_n}))$,

where $A_1,\ldots,A_n$ are the only wff-schemata in $\mathcal{A}$ and $x_{j,1},\ldots,x_{j,i_j}$ are the only variables (i.e. free or bound by $\mathcal{A}$) in A_j and the only <u>free</u> variables (i.e. free or bound by $\mathcal{A}$) in $\mathcal{B}_j$.

<u>Theorems.</u>

D.R.1. $\underline{\vdash_{LS} A \Rightarrow \vdash_{LS} (Ax)A.}$

R.5 : $\vdash_{LS} (p \supset p) \supset A \Rightarrow \vdash_{LS} (p \supset p) \supset (Ax)A$ ______(1)

LS : $(p \supset p) \supset A$ _____(2)

(1), (2) : $(p \supset p) \supset (Ax)A$ ______(3)

(3), LS : $(Ax)A$.

D.R.2. $\underline{\vdash_P \mathcal{A}(A_1(x_{1,1},\ldots,x_{1,i_1}),\ldots, A_n(x_{n,1},\ldots,x_{n,i_n})), \vdash_{LS} C\mathcal{B}_1(x_{1,1},\ldots,x_{1,i_1}),\ldots, \vdash_{LS} C\mathcal{B}_n(x_{n,1},\ldots,x_{n,i_n}) \Rightarrow \vdash_{LS} \mathcal{A}(\mathcal{B}_1(x_{1,1},\ldots,x_{1,i_1}),\ldots,\mathcal{B}_n(x_{n,1},\ldots,x_{n,i_n})).}$

By the completeness of L and the definitions in L of the connective symbols of $\mathcal{A}$, $\vdash_P \mathcal{A}(A_1,\ldots,A_n) \Rightarrow \vdash_L (Ax_{1,j})C\mathcal{B}_1 \supset. \ldots. \supset (Ax_{n,j})C\mathcal{B}_n \supset \mathcal{A}(\mathcal{B}_1,\ldots,\mathcal{B}_n)$, with variables as above. By Rule 6, $\vdash_{LS} (Ax_{1,j})S\mathcal{B}_1 \supset. \ldots. \supset (Ax_{n,j})S\mathcal{B}_n \supset. (Ax_{1,j})C\mathcal{B}_1 \supset. \ldots. \supset (Ax_{n,j})C\mathcal{B}_n \supset \mathcal{A}(\mathcal{B}_1,\ldots,\mathcal{B}_n)$. Since $\vdash_{LS} C\mathcal{B}_1,\ldots, \vdash_{LS} C\mathcal{B}_n$, by LS, $\vdash_{LS} S\mathcal{B}_1,\ldots, \vdash_{LS} S\mathcal{B}_n$. By D.R.1, $\vdash_{LS} (Ax_{1,j})S\mathcal{B}_1,\ldots, \vdash_{LS} (Ax_{n,j})S\mathcal{B}_n$. Hence, $\vdash_{LS} \mathcal{A}(\mathcal{B}_1,\ldots,\mathcal{B}_n)$.

D.R.3. $\underline{\vdash_{LS} A \supset B \Rightarrow \vdash_{LS} (Ax)A \supset (Ax)B.}$

A.12, LS : $(Ax)A \supset B$ ______(1)

(1), R.5 : $(Ax)A \supset (Ax)B$.

1. $\underline{(Ax)A \supset (Sx)A.}$

A.12, A.13, LS : $(Ax)A \supset (Sx)A$.

2. $\underline{(Ax)TA(x) \supset T(Ax)A(x).}$

L : $(Ax)TA(x) \supset T(Ax)A(x)$ ______ (1)

(1), R.6 : $(Ax)SA(x) \supset. (Ax)TA(x) \supset T(Ax)A(x)$ ______ (2)

(2), LS : $(Ax)SA(x) \ \& \ (Ax)TA(x) \supset T(Ax)A(x)$ ______ (3)

LS : $TA(x) \supset SA(x)$ ______ (4)

(4), D.R.3 : $(Ax)TA(x) \supset (Ax)SA(x)$ ______ (5)

(3), (5), LS : $(Ax)TA(x) \supset T(Ax)A(x)$.

3. $\underline{(Ax)(TA(x) \ v \ PA(x)) \ \& \ (Sx)PA(x) \supset P(Ax)A(x).}$

L : T.3 ______ (1)

(1), R.6 : $(Ax)SA(x) \supset$ T.3 ______ (2)

(2), LS : $(Ax)SA(x) \ \& \ (Ax)(TA(x) \ v \ PA(x)) \ \& \ (Sx)PA(x) \supset P(Ax)A(x)$

______ (3)

LS : $TA(x) \ v \ PA(x) \supset SA(x)$ ______ (4)

(4), D.R.3 : $(Ax)(TA(x) \ v \ PA(x)) \supset (Ax)SA(x)$ ______ (5)

(3), (5), LS : T.3.

4. $\underline{(Ax)SA(x) \ \& \ (Sx)FA(x) \supset F(Ax)A(x).}$

L : T.4 ______ (1)

(1), R.6 : $(Ax)SA(x) \supset$ T.4 ______ (2)

(2), LS : T.4.

5. $\underline{T(Sx)A(x) \supset (Sx)TA(x).}$

LS : $T(Sx)A(x) \supset \sim P(Sx)A(x)$ ______ (1)

P : $((Ax)A(x) \ \& \ (Sx)B(x) \supset C) \supset (\sim C \supset \sim(Ax)A(x) \ v \ \sim(Sx)B(x))$ ____ (2)

LS, D.R.1 : $(Ax)C\sim TA(x)$ ______ (3)

LS, D.R.1 : $(Ax)CPA(x)$ ______ (4)

LS : $CP(Sx)A(x)$ ______(5)

(2), (3), (4), (5), D.R.2 : $((Ax){\sim}TA(x)\ \&\ (Sx)PA(x) \supset P(Sx)A(x))$

$\supset ({\sim}P(Sx)A(x) \supset {\sim}(Ax){\sim}TA(x) \vee {\sim}(Sx)PA(x))$ ______(6)

A.16, (6) : ${\sim}P(Sx)A(x) \supset {\sim}(Ax){\sim}TA(x) \vee {\sim}(Sx)PA(x)$ ______(7)

(1), (7), LS : $T(Sx)A(x) \supset {\sim}(Ax){\sim}TA(x) \vee {\sim}(Sx)PA(x)$ ______(8)

P : ${\sim}(Ax){\sim}A(x) \supset (Sx)A(x)$ ______(9)

LS, D.R.1 : $(Ax)CTA(x)$ ______(10)

(9), (10), D.R.2 : ${\sim}(Ax){\sim}TA(x) \supset (Sx)TA(x)$ ______(11)

(8), (11), LS : $T(Sx)A(x) \supset (Sx)TA(x) \vee {\sim}(Sx)PA(x)$ ______(12)

Similarly, $T(Sx)A(x) \supset (Sx)({\sim}FA(x)\ \&\ SA(x)) \vee (Ax){\sim}FA(x)$, ______(13)

using A.17.

Similarly, $T(Sx)A(x) \supset (Sx)SA(x)$, ______(14) using A.18.

P : $((Sx)A(x)\ \&\ (Ax)(B(x) \vee C(x) \vee {\sim}A(x))) \supset (Sx)(B(x) \vee C(x))$

______(15)

LS, D.R.1, D.R.2, (15) : $((Sx)SA(x)\ \&\ (Ax)(TA(x) \vee PA(x) \vee {\sim}SA(x)))$

$\supset (Sx)(TA(x) \vee PA(x))$ ______(16)

LS, D.R.3 : $(Ax){\sim}FA(x) \supset (Ax)(TA(x) \vee PA(x) \vee {\sim}SA(x))$ ______(17)

(16), (17), LS : $(Ax){\sim}FA(x)\ \&\ (Sx)SA(x) \supset (Sx)(TA(x) \vee PA(x))$

______(18)

(13), (14), LS : $T(Sx)A(x) \supset ((Ax){\sim}FA(x)\ \&\ (Sx)SA(x)) \vee ((Sx)({\sim}F$

$A(x)\ \&\ SA(x))\ \&\ (Sx)SA(x))$ ______(19)

P : $(Ax)(A(x) \supset B(x)) \supset. (Sx)A(x) \supset (Sx)B(x)$ ______(20)

(20), LS, D.R.1, D.R.2 : $(Ax)({\sim}FA(x)\ \&\ SA(x) \supset TA(x) \vee PA(x))$

$\supset. (Sx)({\sim}FA(x)\ \&\ SA(x)) \supset (Sx)(TA(x) \vee PA(x))$ ______(21)

LS, D.R.1 : $(Sx)(\sim FA(x) \& SA(x)) \supset (Sx)(TA(x) \vee PA(x))$ ______(22)

(19), (22), LS : $T(Sx)A(x) \supset ((Ax)\sim FA(x) \& (Sx)SA(x)) \vee ((Sx)(TA(x) \vee PA(x)) \& (Sx)SA(x))$ ______(23)

(18), (23), LS : $T(Sx)A(x) \supset (Sx)(TA(x) \vee PA(x))$ ______(24)

(24), P, D.R.2 : $T(Sx)A(x) \supset (Sx)TA(x) \vee (Sx)PA(x)$ ______(25)

(12), (25), P, D.R.2 : $T(Sx)A(x) \supset (Sx)TA(x)$.

6. <u>$S(Sx)A(x) \supset (Sx)SA(x)$.</u>

P : $((Ax)\sim A(x) \supset \sim B(x)) \supset (B(x) \supset (Sx)A(x))$ ______(1)

(1), LS, D.R.1, D.R.2 : $((Ax)\sim SA(x) \supset \sim S(Sx)A(x)) \supset (S(Sx)A(x) \supset (Sx)SA(x))$ ______(2)

A.18, (2) : $S(Sx)A(x) \supset (Sx)SA(x)$.

7. <u>$S(Ax)A(x) \supset (Ax)SA(x)$.</u>

P : $((Sx)\sim A(x) \supset \sim B(x)) \supset (B(x) \supset (Ax)A(x))$ ______(1)

(1), LS, D.R.1, D.R.2 : $((Sx)\sim SA(x) \supset \sim S(Ax)A(x)) \supset (S(Ax)A(x) \supset (Ax)SA(x))$ ______(2)

A.14, (2) : $S(Ax)A(x) \supset (Ax)SA(x)$.

<u>Meta-theorem 14.</u>

The Deduction Theorem holds for $\supset$, i.e. if $A_1,\dots,A_n \vdash_{LS} B$ then $A_1,\dots,A_{n-1} \vdash_{LS} A_n \supset B$, provided Rule 5 is not used to generalise on any variable of A_n and Rule 4 is not used to change a free variable of A_n.

<u>Proof.</u> The proof is the same as that of Meta-theorem 1, except that the systems are L and LS instead of P and S.

Meta-theorem 15.

If B is valid according to the 4-valued matrices and the properties stated for A and S, then B is a thesis of the axiomatic system LS.

Proof. The proof is similar to that of Meta-theorem 2. I will indicate the differences between that proof and this.

In showing that $\Gamma_n^{m\ 1}$ is consistent if Γ_n^m is, the following theorems and derived rules of LS are required :

$\vdash_{LS} A \supset B,\ \vdash_{LS} A \supset \sim B \Rightarrow \vdash_{LS} \sim TA.$

$\vdash_{LS} A(x) \Rightarrow \vdash_{LS} (Ax)A(x).$ (D.R.1)

$(Ax)\sim TA(x) \supset \sim(Sx)TA(x).$ (P, D.R.2)

$\vdash_{LS} A \Rightarrow \vdash_{LS} TA.$

$T(Sx)A(x) \supset (Sx)TA(x).$ (T.5)

The property (b) of Γ_ω is :

If $CA \in \Gamma_\omega$ and $A \notin \Gamma_\omega$ then $\sim A \in \Gamma_\omega$. To show that this property holds, we need the following :

$\vdash_{LS} CA,\ \vdash_{LS} \sim TA \Rightarrow \vdash_{LS} FA.$

$\vdash_{LS} TA \Rightarrow \vdash_{LS} A.$

The property (d) of Γ_ω is : At least one of TA, PA, FA, ~SA is a member of Γ_ω. To prove this we need the following : CTA, CPA, CFA, C~SA, TA v PA v FA v ~SA, all theorems of LS.

The property (e) of Γ_ω is : At most one of TA, PA, FA, ~SA is a member of Γ_ω. To prove this, we need the following : TA ⊃ ~FA, TA ⊃ ~PA, TA ⊃ SA, PA ⊃ ~TA, PA ⊃ ~FA, PA ⊃ SA, FA ⊃ ~TA, FA ⊃ ~PA, FA ⊃ SA, ~SA ⊃ ~TA, ~SA ⊃ ~PA, ~SA ⊃ ~FA, all theorems of LS .

For the assignment of values, if $PA \in \Gamma_\omega$ then A has P. $G(x_1,\ldots,x_n)$ has F if G has S and $FG(a_1,\ldots,a_n) \in \Gamma_\omega$ for some choice of $a_1,\ldots,a_n$. If G does not have F and does not have T but has S then G has P. G has ~S if G does not have S.

In checking the connectives ~, &, $\supset$, and T_n, one needs the following : $TA \supset F{\sim}A$, $PA \supset P{\sim}A$, $FA \supset T{\sim}A$, ${\sim}SA \supset {\sim}S{\sim}A$, $TA \mathbin{\&} TB \supset T(A \supset B)$, $TA \mathbin{\&} PB \supset P(A \supset B)$, $TA \mathbin{\&} FB \supset F(A \supset B)$, $TA \mathbin{\&} {\sim}SB \supset {\sim}S(A \supset B)$, $PA \vee FA \vee {\sim}SA \supset T(A \supset B)$, $TA \mathbin{\&} TB \supset T(A \mathbin{\&} B)$, $(TA \mathbin{\&} PB) \vee (PA \mathbin{\&} PB) \vee (PA \mathbin{\&} TB) \supset P(A \mathbin{\&} B)$, $(FA \mathbin{\&} SB) \vee (SA \mathbin{\&} FB) \supset F(A \mathbin{\&} B)$, ${\sim}SA \vee {\sim}SB \supset {\sim}S(A \mathbin{\&} B)$, $TA \supset TT_nA$, $PA \vee FA \vee {\sim}SA \supset {\sim}ST_nA$, all theorems of LS.

In checking the quantifier, A, in part (a) one needs the following : ${\sim}T(Ax)A(x) \supset {\sim}(Ax)TA(x)$, ${\sim}(Ax)TA(x) \supset (Sx){\sim}TA(x)$, both of which are theorems of LS (T.2, P, D.R.2 ; P, D.R.2).

In part (b), one needs the following : $A \supset (Sx)A$, $(Sx){\sim}SA(x) \supset {\sim}(Ax)SA(x)$, ${\sim}(Ax)SA(x) \supset {\sim}S(Ax)A(x)$, all of which are theorems of LS (A.13 ; P, D.R.2 ; T.7, P, D.R.2).

In part (c), one needs the following : $A \supset (Sx)A$, $C(Ax)SA(x)$, ${\sim}(Ax)SA(x) \supset (Sx){\sim}SA(x)$, $(Ax)SA(x) \mathbin{\&} (Sx)FA(x) \supset F(Ax)A(x)$, all of which are theorems of LS (A.13 ; P, D.R.2 ; P, D.R.2 ; T.4).

Part (d) becomes :

Let $\Gamma_\omega \vdash PB(w_{m,n})$ for some $w_{m,n}$ and $\Gamma_\omega \vdash TB(w_{m,n})$ or $\Gamma_\omega \vdash PB(w_{m,n})$ for all $w_{m,n}$. Hence $\Gamma_\omega \vdash (Sx)PB(x)$ and $\Gamma_\omega \vdash TB(w_{m,n}) \vee PB(w_{m,n})$, for all $w_{m,n}$. Let $(Ax)(TB(x) \vee PB(x)) \notin \Gamma_\omega$. By P and D.R.2,

$\Gamma_\omega \vdash C(Ax)(TB(x) \vee PB(x))$. Hence $\Gamma_\omega \vdash \sim(Ax)(TB(x) \vee PB(x))$. By P, D.R.2, $\Gamma_\omega \vdash (Sx)(\sim TB(x) \,\&\, \sim PB(x))$. Hence, $\sim TB(w_{m,n}) \,\&\, \sim PB(w_{m,n}) \in \Gamma_\omega$. By the consistency of Γ_ω, $(Ax)(TB(x) \vee PB(x)) \notin \Gamma_\omega$ and by T.3, $P(Ax)B(x) \in \Gamma_\omega$. Hence, if $B(x)$ has P for some x and T or P for all x then $(Ax)B(x)$ has P.

In checking the quantifier, S, in part (a) one needs the following : $A \supset (Sx)A$, $(Sx)TA(x) \supset T(Sx)A(x)$, both of which are theorems of LS (A.13 ; A.15).

In part (b), one needs the following : $C(Ax){\sim}SA(x)$, $\sim(Ax){\sim}SA(x) \supset (Sx)SA(x)$, $(Ax){\sim}SA(x) \supset {\sim}S(Sx)A(x)$. (P, D.R.2 ; P, D.R.2 ; A.18).

Part (c) becomes :

Let $\Gamma_\omega \vdash FB(w_{m,n})$ or $\Gamma_\omega \vdash \sim SB(w_{m,n})$ for all $w_{m,n}$, and $\Gamma_\omega \vdash FB(w_{m,n})$ for some $w_{m,n}$. Hence $\Gamma_\omega \vdash (Sx)FB(x)$ and $\Gamma_\omega \vdash FB(w_{m,n}) \vee \sim SB(w_{m,n})$ for all $w_{m,n}$. Let $(Ax)(FB(x) \vee \sim SB(x)) \notin \Gamma_\omega$. Then, since by P and D.R.2, $\Gamma_\omega \vdash C(Ax)(FB(x) \vee \sim SB(x))$, $\sim(Ax)(FB(x) \vee \sim SB(x)) \in \Gamma_\omega$. By P and D.R.2, $(Sx)(\sim FB(x) \,\&\, SB(x)) \in \Gamma_\omega$. Hence $\sim FB(w_{m,n}) \,\&\, SB(w_{m,n}) \in \Gamma_\omega$. By the consistency of Γ_ω, $(Ax)(FB(x) \vee \sim SB(x)) \in \Gamma_\omega$ and by A.17, $F(Sx)B(x) \in \Gamma_\omega$. Hence, if $B(x)$ has F for some x and has F or ~S for all x then $(Sx)B(x)$ has F.

Part (d) becomes :

Let $\Gamma_\omega \vdash PB(w_{m,n})$ for some $w_{m,n}$ and let $TB(w_{m,n}) \notin \Gamma_\omega$ for all $w_{m,n}$. Then $\Gamma_\omega \vdash (Sx)PB(x)$. Let $(Ax){\sim}TB(x) \notin \Gamma_\omega$. Since, by P and D.R.2, $C(Ax){\sim}TB(x)$, $\sim(Ax){\sim}TB(x) \in \Gamma_\omega$. By P and D.R.2, $\sim(Ax){\sim}TB(x) \supset (Sx)TB(x)$, and hence $(Sx)TB(x) \notin \Gamma_\omega$ and $TB(w_{m,n}) \notin \Gamma_\omega$ for some $w_{m,n}$. By the consistency of Γ_ω, $(Ax){\sim}TB(x) \in \Gamma_\omega$. By A.16, $P(Sx)B(x) \in \Gamma_\omega$. Hence, if $B(x)$ has P for some x and does not have T for all x then (Sx) $B(x)$ has P.

The completeness of the axiomatic system LS can now be shown as can its consistency, which follows by the same procedure as used in Meta-theorem 2.

<u>Meta-theorem 16.</u>

Substitutivity of Equivalents holds in LS for $\leftrightarrow$, i.e. if $\vdash_{LS}$ A $\leftrightarrow$ B then $\vdash_{LS}$ $C(A) \leftrightarrow C(B)$, where substitution into C can be made for any argument place.

<u>Proof.</u> Since $\vdash_{LS} A \leftrightarrow B \Rightarrow \vdash_{LS} (Ax)A \leftrightarrow (Ax)B$ and $\vdash_{LS} (Sx)A \leftrightarrow (Sx)B$, the theorem follows as before.

<u>Meta-theorem 17.</u>

Let B be a wff of LS containing only the connectives, $\supset$, T_n, T, &, v, ~T and the quantifiers, A, S. Let A be a wff of P obtained from B by deleting any T's or T_n's and replacing ~T by ~. Then, if $\vdash_P$ A then $\vdash_{LS}$ B.

<u>Proof.</u> The proof is the same as that of Meta-theorem 15 except that, instead of there being four values, T, P, F, and ~S, there are two values, T and ~T. All the necessary theorems are available because of the completeness of the system LS.

<u>Meta-theorem 18.</u>

Let C be a wff of LS containing only the connectives, $\supset$, T_n, T,

&, v, ~T and quantifiers, A, S. Then, if $\vdash_{LS} A \equiv B$ then $\vdash_{LS} C(A) \equiv C(B)$, where substitution into C can be made for any argument place.

<u>Proof.</u> Since $\vdash_{LS} A \equiv B \Rightarrow \vdash_{LS} (Ax)A \equiv (Ax)B$ and $\vdash_{LS} (Sx)A \equiv (Sx)B$, the theorem follows as before.

<u>Meta-theorem 19.</u>

If the domain is restricted to all x's such that D(x) is true, then the axioms and rules restricted to this domain will still hold, provided that the domain is non-empty, i.e. $\vdash_{LS} (Sx)D(x)$.

<u>Proof.</u> The variables are restricted as follows : $(Sx)(T_n D(x)$ & $A(x))$ is the restriction of $(Sx)A(x)$ to $D(x)$. $(Ax)(D(x) \supset A(x))$ is the restriction of $(Ax)A(x)$ to $D(x)$. The proof follows the lines of that of Meta-theorem 6.

CHAPTER 3.

THE THEORY OF INDIVIDUALS.

Firstly, I would like to clarify my concept of an individual. It is said on p.45 of [16], "What is conceived as an individual and what as a class is thus relative to the discourse within which the conception occurs. One task of applied logic is to determine which entities are to be construed as individuals and which as classes when the purpose is the development of a comprehensive systematic discourse." Here I am presenting a comprehensive system which includes all kinds of individuals, although interpretations of it can be restricted to particular contexts as will occur in the model I give for it which will contain only one individual. Lafleur, in [14], criticizes this comprehensive system saying, "Unfortunately, there are no entities which are discrete in this total sense : any two named will have many characteristics in common, especially if both are "concrete"." Lafleur is mistaken because he thinks that the mere existence of some common characteristic will ensure that two individuals will have a common part. Two material objects which are spatially separate at a given time will be discrete, that is, they will have no common parts. The common characteristics needed to ensure that two individuals have a common part depend on the nature of the individuals. Two material objects having a common material part at a given time, two periods of time having a common temporal part, two groups (that is, individual fusions as opposed to classes) of people with a common person and two individual fusions of a person's thoughts containing the same thought (or, indeed, any two individual fusions containing the same individual) all have a common part. Two individuals of different kinds will not have any common part.

It is useful to introduce the notion of atomic individual, which is one having no individual as a proper part of it; for example, a point in time, ~~the smallest particle of atomic physics and the colour of a particular atom or molecule at a particular point in time are all atomic individuals.~~

Unlike most class theories, there is a universal individual, which is defined as the fusion of all individuals. Every individual is a part of the universal individual.

Wang, in his paper, [32], criticizes Goodman's calculus of individuals because he does not commit himself on the question of the total number of individuals. The usual sorts of axiom of infinity, like 'every individual has a proper part' and 'every individual is a proper part of some individual', must be rejected because of the existence of atomic individuals and a universal individual. Since points in time and space are taken as individuals there must be at least $2^{\aleph_0}$ individuals. Periods of time and volumes of space are individual fusions of these points and are hence fusions of $2^{\aleph_0}$ atomic individuals.

Although I use Goodman's calculus of individuals and accept some of his notions of individual, I admit some individuals, e.g. points and some idealized concepts, which he does not admit and I take the notion of individual as absolute rather than "relative to the discourse within which the conception occurs." I say that an entity is either a class or a non-class and regard any non-class as an individual, whether it is a material object or an idealized concept.

For the purpose of formalising the calculus of individuals a theory of classes or sets must be assumed so that fusions of all the individual members of certain classes can be formed. In the next chapter I add individuals to the class theory, NBG, and use individuals as well as the null set to generate classes. I regard the class of all individuals as a set because there is no reason why it should be a proper class and no paradoxes arise out of its being a set. I will present a 2-valued calculus of individuals which can later be fitted into 3 and 4-valued theories of classes and individuals. For this calculus, I must introduce sets with individual members only as well as the individuals themselves.

The formal axiomatic system is as follows :

Primitives.

1. k,l,m,n,.......(individual variables).

2. f',g',h',........(variables over sets with individual members only).

3. o ('overlaps'), $\in$ ('is a member of').

4. $\sim$, & (connectives), A (quantifier).

Formation Rules.

1. If k and l are individual variables then kol is a wff.

2. If k is an individual variable and f' is a set variable then $k \in f'$ is a wff.

3. If A and B are wffs then $\sim A$ and A & B are wffs.

4. If A is a wff and k is an individual variable then (Ak)A is a wff.

5. If A is a wff and f' is a set variable then (Af')A is a wff.

Definitions.

1. $k \leq l =_{df} (Am)(mok \supset mol)$. (k is part of l.)

2. $k = l =_{df} (Am)(mok \equiv mol)$. (k is identical with l.)

3. $k < l =_{df} k \leq l \,\&\, \sim(k = l)$. (k is a proper part of l.)

4. $k \rceil\!\lfloor l =_{df} \sim(kol)$. (k is discrete from l.)

5. $kFuf' =_{df} (Am)(m \rceil\!\lfloor k \equiv (Al)(l \in f' \supset m \rceil\!\lfloor l))$. (k is the fusion of f'.)

6. $kNuf' =_{df} (Am)(m \leq k \equiv (Al)(l \in f' \supset m \leq l))$. (k is the nucleus of f'.)

7. $f' \subseteq g' =_{df} (Ak)(k \in f' \supset k \in g')$. (f' is a subset of g'.)

8. $f' = g' =_{df} (Ak)(k \in f' \equiv k \in g')$. (f' is identical with g'.)

9. $f' \subset g' =_{df} f' \subseteq g' \,\&\, \sim(f' = g')$. (f' is a proper subset of g'.)

Axioms.

1. $kol \equiv (Sm)(An)(nom \supset nok \,\&\, nol)$.

2. $(Sk)(k \in f') \supset (Sl)(lFuf')$.

3. $(Sf')(Ak)(k\epsilon f' \equiv \emptyset(k,l_1,\ldots,l_m,f_1',\ldots,f_n'))$, where $\emptyset$ is constructed using ϵ, o, ~, & and A, where the quantification can be over individual or set variables.

4. $k=l \supset . k\epsilon f' \supset l\epsilon f'$.

Theorems.

1. $k=l \equiv . k \leqslant l \;\&\; l \leqslant k$.

P : $(Am)(mok \equiv mol) \equiv . (Am)(mok \supset mol) \;\&\; (Am)(mol \supset mok)$ ______(1)

(1), Defns.1 and 2 : $k=l \equiv . k \leqslant l \;\&\; l \leqslant k$.

2. $kol \equiv (Sm)(m \leqslant k \;\&\; m \leqslant l)$.

P : $(An)(nom \supset nok \;\&\; nol) \equiv . (An)(nom \supset nok) \;\&\; (An)(nom \supset nol)$ ______(1)

(1), A.1 : $kol \equiv (Sm)((An)(nom \supset nok) \;\&\; (An)(nom \supset nol))$ ______(2)

(2), Defn.1 : $kol \equiv (Sm)(m \leqslant k \;\&\; m \leqslant l)$.

3. $k \leqslant k$.

P : $(Am)(mok \supset mok)$ ______(1)

(1), Defn.1 : $k \leqslant k$.

4. kok.

T.3 : $k \leqslant k \;\&\; k \leqslant k$ ______(1)

(1), P : $(Sm)(m \leqslant k \;\&\; m \leqslant k)$ ______(2)

(2), T.2 : kok.

5. $k \leqslant l \supset kol$.

P : $(Am)(mok \supset mol) \supset . kok \supset kol$ ______(1)

(1), T.4, P : $(Am)(mok \supset mol) \supset kol$ ______(2)

(2), Defn.1 : $k \leqslant l \supset kol$.

6. $\underline{kol \equiv lok.}$

P : $(Sm)(m \leq k \ \& \ m \leq l) \equiv (Sm)(m \leq l \ \& \ m \leq k)$ ______(1)

(1), T.2 : $kol \equiv lok.$

7. $\underline{k=k.}$

P : $(Am)(mok \equiv mok)$ ______(1)

(1), Defn.2 : $k=k.$

8. $\underline{k=l \equiv l=k.}$

P : $(Am)(mok \equiv mol) \equiv (Am)(mol \equiv mok)$ _______(1)

(1), Defn.2 : $k=l \equiv l=k.$

9. $\underline{k=l \ \& \ l=m \supset k=m.}$

P : $(An)(nok \equiv nol) \ \& \ (An)(nol \equiv nom) \supset (An)(nok \equiv nom)$ ______(1)

(1), Defn.2 : $k=l \ \& \ l=m \supset k=m.$

10. $\underline{k \leq l \ \& \ l \leq m \supset k \leq m.}$

P : $(An)(nok \supset nol) \ \& \ (An)(nol \supset nom) \supset (An)(nok \supset nom)$ _____(1)

(1), Defn.1 : $k \leq l \ \& \ l \leq m \supset k \leq m.$

11. $\underline{k=l \supset k \leq l.}$

P : $(Am)(mok \equiv mol) \supset (Am)(mok \supset mol)$ _______(1)

(1), Defns.1 and 2 : $k=l \supset k \leq l.$

12. $\underline{k \leq l \equiv . k < l \ v \ k=l.}$

T.11, P : $k \leq l \equiv k \leq l \ v \ k=l$ ______(1)

(1), P : $k \leq l \equiv (k \leq l \ \& \ \sim(k=l)) \ v \ k=l$ _______(2)

(2), Defn.3 : $k \leq l \equiv k < l \ v \ k=l.$

13. $\underline{k < l \supset \sim(l \leq k).}$

P : $l \leq k \supset \sim(k \leq l) \ v \ l \leq k$ ______(1)

(1), P : $1 \leq k \supset \sim(k \leq 1) \vee (k \leq 1 \,\&\, 1 \leq k)$ ______(2)

(2), T.1 : $1 \leq k \supset \sim(k \leq 1) \vee k=1$ ______(3)

(3), P, Defn.3 : $1 \leq k \supset \sim(k < 1)$ ______(4)

(4), P : $k < 1 \supset \sim(1 \leq k)$.

14. $\underline{k < 1 \supset \sim(1 < k)}$.

T.12, P : $1 < k \supset 1 \leq k$ ______(1)

(1), P : $\sim(1 \leq k) \supset \sim(1 < k)$ ______(2)

(2), T.13, P : $k < 1 \supset \sim(1 < k)$.

15. $\underline{\sim(k < k)}$.

P : $\sim(k \leq k \,\&\, \sim(k=k))$ ______(1)

(1), Defn.3 : $\sim(k < k)$.

16. $\underline{k < 1 \,\&\, 1 \leq m \supset k < m}$.

Defn.3, P : $k < 1 \supset k \leq 1$ ______(1)

(1), T,10, P : $k < 1 \,\&\, 1 \leq m \supset k \leq m$ ______(2)

T.13 : $k < 1 \supset \sim(1 \leq k)$ ______(3)

T.10, P : $1 \leq m \,\&\, \sim(1 \leq k) \supset \sim(m \leq k)$ ______(4)

(3), (4), P : $k < 1 \,\&\, 1 \leq m \supset \sim(m \leq k)$ ______(5)

T.11, P : $\sim(m \leq k) \supset \sim(m=k)$ ______(6)

T.8, P, (5), (6) : $k < 1 \,\&\, 1 \leq m \supset \sim(k=m)$ ______(7)

(2), (7), P : $k < 1 \,\&\, 1 \leq m \supset k \leq m \,\&\, \sim(k=m)$ ______(8)

(8), Defn.3 : $k < 1 \,\&\, 1 \leq m \supset k < m$.

17. $\underline{k \leq 1 \equiv (Am)(m \leq k \supset m \leq 1)}$.

P : $(Am)(m \leq k \supset m \leq 1) \supset. k \leq k \supset k \leq 1$ ______(1)

(1), T.3, P : $(Am)(m \leq k \supset m \leq 1) \supset k \leq 1$ ______(2)

T.10, P : $k \lesssim l \supset.\ m \lesssim k \supset m \lesssim l$ _______ (3)

(3), P : $k \lesssim l \supset (Am)(m \lesssim k \supset m \lesssim l)$ _______ (4)

(2), (4), P : $k \lesssim l \equiv (Am)(m \lesssim k \supset m \lesssim l)$.

18. $\underline{k=l \equiv (Am)(m \lesssim k \equiv m \lesssim l).}$

T.17, P : $k \lesssim l \ \&\ l \lesssim k \equiv (Am)(m \lesssim k \supset m \lesssim l) \ \&\ (Am)(m \lesssim l \supset m \lesssim k)$

_______ (1)

(1), T.1, P : $k=l \equiv (Am)(m \lesssim k \equiv m \lesssim l)$.

19. $\underline{\sim(k \urcorner\!\llcorner k).}$

T.4, Defn.4 : $\sim(k \urcorner\!\llcorner k)$.

20. $\underline{k \urcorner\!\llcorner l \equiv l \urcorner\!\llcorner k.}$

T.6, Defn.4, P : $k \urcorner\!\llcorner l \equiv l \urcorner\!\llcorner k$.

21. $\underline{k \lesssim l \equiv (Am)(m \urcorner\!\llcorner l \supset m \urcorner\!\llcorner k).}$

Defns.1 and 4, P : $k \lesssim l \equiv (Am)(m \urcorner\!\llcorner l \supset m \urcorner\!\llcorner k)$.

22. $\underline{k=l \equiv (Am)(m \urcorner\!\llcorner k \equiv m \urcorner\!\llcorner l).}$

Defns.2 and 4, P : $k=l \equiv (Am)(m \urcorner\!\llcorner k \equiv m \urcorner\!\llcorner l)$.

23. $\underline{(Af')(k \in f' \equiv l \in f') \supset k=l.}$

Hyp : $(Af')(k \in f' \equiv l \in f')$ _______ (1)

A.3 : $(Sf')(Ak)(k \in f' \equiv mok)$ _______ (2)

(2), P : $k \in f_m' \equiv mok$ _______ (3) (f_m' is a constant.)

(2), P : $l \in f_m' \equiv mol$ _______ (4)

(1), (3), (4), P : $mok \equiv mol$ _______ (5)

(5), P : $(Am)(mok \equiv mol)$ _______ (6)

(6), Defn.2 : $k=l$ _______ (7)

(1), (7), P : $(Af')(k \in f' \equiv l \in f') \supset k=l$.

24. $k=l \equiv (Af')(k\in f' \equiv l\in f')$.

A.4, T.8, P : $k=l \supset . k\in f' \equiv l\in f'$ ______ (1)

(1), P : $k=l \supset (Af')(k\in f' \equiv l\in f')$ ______ (2)

T.23, (2), P : $k=l \equiv (Af')(k\in f' \equiv l\in f')$.

25. $k=l \supset . \phi(k) \equiv \phi(l)$, for any wff ϕ.

Defn.2, P : $k=l \supset . mok \equiv mol$ ______ (1)

(1), T.6, P : $k=l \supset . kom \equiv lom$ ______ (2)

T.24, P : $k=l \supset . k\in f' \equiv l\in f'$ ______ (3)

Using (1), (2) and (3), by induction on the number of connectives and quantifiers of ϕ, $k=l \supset . \phi(k) \equiv \phi(l)$ can be shown.

26. $(Sk)(k\in f') \supset (S!l)(lFuf')$.

A.2 : $(Sk)(k\in f') \supset (Sl)(lFuf')$ ______ (1)

Defn.5 : $lFuf' \equiv (An)(n \rceil\!\lfloor l \equiv (An')(n'\in f' \supset n \rceil\!\lfloor n'))$ ______ (2)

Defn.5 : $mFuf' \equiv (An)(n \rceil\!\lfloor m \equiv (An')(n'\in f' \supset n \rceil\!\lfloor n'))$ ______ (3)

(2), (3), P : $lFuf' \ \& \ mFuf' \supset . (n \rceil\!\lfloor l \equiv (An')(n'\in f' \supset n \rceil\!\lfloor n')) \ \&$
$(n \rceil\!\lfloor m \equiv (An')(n'\in f' \supset n \rceil\!\lfloor n'))$ ______ (4)

(4), P : $lFuf' \ \& \ mFuf' \supset . n \rceil\!\lfloor l \equiv n \rceil\!\lfloor m$ ______ (5)

(5), P, T.22 : $lFuf' \ \& \ mFuf' \supset l=m$ ______ (6)

(1), (6), P : $(Sk)(k\in f') \supset (S!l)(lFuf')$.

Under the condition that f' is non-empty, we can now introduce the symbol $Fu^{\iota}f'$ for the unique l such that lFuf'.

27. $(Sk)(k\in f') \supset (Am)(m \rceil\!\lfloor Fu^{\iota}f' \equiv (Al)(l\in f' \supset m \rceil\!\lfloor l))$.

Defn.Fu f' : $(Sk)(k\in f') \supset (Fu^{\iota}f')Fuf'$ ______ (1)

(1), Defn.5, P : $(Sk)(k\in f') \supset (Am)(m \rceil\!\lfloor Fu^{\iota}f' \equiv (Al)(l\in f' \supset m \rceil\!\lfloor l))$.

28. $(Sk)(k \in f') \supset . f' \subseteq g' \supset Fu\text{‘}f' \leq Fu\text{‘}g'$.

Hyp : $(Sk)(k \in f')$ ______(1)

Hyp : $f' \subseteq g'$ ______(2)

(1), (2), Defn. 7, P : $(Sk)(k \in g')$ ______(3)

(1), T.27 : $(Am)(m \rceil\!\lfloor Fu\text{‘}f' \equiv (Al)(l \in f' \supset m \rceil\!\lfloor l))$ ______(4)

(3), T.27 : $(Am)(m \rceil\!\lfloor Fu\text{‘}g' \equiv (Al)(l \in g' \supset m \rceil\!\lfloor l))$ ______(5)

(2), Defn. 7, P : $l \in g' \supset m \rceil\!\lfloor l \supset . l \in f' \supset m \rceil\!\lfloor l$ ______(6)

(4), (5), (6), P : $m \rceil\!\lfloor Fu\text{‘}g' \supset m \rceil\!\lfloor Fu\text{‘}f'$ ______(7)

(7), P, T.21 : $Fu\text{‘}f' \leq Fu\text{‘}g'$ ______(8)

(1), (2), (8), P : $(Sk)(k \in f') \supset . f' \subseteq g' \supset Fu\text{‘}f' \leq Fu\text{‘}g'$.

29. $(Sk)(k \in f') \supset . f' = g' \supset Fu\text{‘}f' = Fu\text{‘}g'$.

Hyp : $(Sk)(k \in f')$ ______(1)

Hyp : $f' = g'$ ______(2)

(1), (2), Defn. 8, P : $(Sk)(k \in g')$ ______(3)

Defns. 7 and 8, P : $f' = g' \equiv f' \subseteq g' \;\&\; g' \subseteq f'$ ______(4)

(1), (4), T.8, P : $Fu\text{‘}f' \leq Fu\text{‘}g' \;\&\; Fu\text{‘}g' \leq Fu\text{‘}f'$ ______(5)

(5), T.1 : $Fu\text{‘}f' = Fu\text{‘}g'$ ______(6)

(1), (2), (6), P : $(Sk)(k \in f') \supset . f' = g' \supset Fu\text{‘}f' = Fu\text{‘}g'$.

30. $(S!f')(Ak)(k \in f' \equiv \phi(k))$, where ϕ is any wff.

A.3 : $(Sf')(Ak)(k \in f' \equiv \phi(k))$ ______(1)

P : $(Ak)(k \in f' \equiv \phi(k)) \;\&\; (Ak)(k \in g' \equiv \phi(k)) \supset (Ak)(k \in f' \equiv k \in g')$

______(2)

(1), (2), Defn. 8, P : $(S!f')(Ak)(k \in f' \equiv \phi(k))$.

We can now introduce the symbol $\{k : \phi(k)\}$ for the unique f' such that $(Ak)(k \in f' \equiv \phi(k))$.

31. $(Ak)(k \in \{k : \phi(k)\} \equiv \phi(k))$, for any wff ϕ.

Defn. $\{k : \phi(k)\} : (Ak)(k \in \{k : \phi(k)\} \equiv \phi(k))$.

Define $\{k,l\}$ as $\{m : m=k \vee m=l\}$.

Define $k+l$ as $Fu' \{k,l\}$. [$(Sm)(m \in \{k,l\})$ always holds because $k=k \vee k=l$ holds.]

32. $m \rceil\!\lfloor (k+l) \equiv m \rceil\!\lfloor k \;\&\; m \rceil\!\lfloor l$.

Defn.+, T.27 : $(Am)(m \rceil\!\lfloor k+l \equiv (An)(n \in \{k,l\} \supset m \rceil\!\lfloor n))$ ______ (1)

(1), Defn. $\{k,l\}$, T.31 : $(Am)(m \rceil\!\lfloor k+l \equiv (An)(n=k \vee n=l \supset m \rceil\!\lfloor n))$ __________ (2)

(2), P : $(Am)(m \rceil\!\lfloor k+l \equiv (An)(n=k \supset m \rceil\!\lfloor n) \;\&\; (An)(n=l \supset m \rceil\!\lfloor n))$ _____ (3)

T.7, P : $(An)(n=k \supset m \rceil\!\lfloor n) \supset m \rceil\!\lfloor k$ _______ (4)

T.22, P : $m \rceil\!\lfloor k \supset (An)(n=k \supset m \rceil\!\lfloor n)$ ________ (5)

(3), (4), (5), P : $(Am)(m \rceil\!\lfloor k+l \equiv m \rceil\!\lfloor k \;\&\; m \rceil\!\lfloor l)$ ________ (6)

(6), P : $m \rceil\!\lfloor (k+l) \equiv m \rceil\!\lfloor k \;\&\; m \rceil\!\lfloor l$.

33. $k+k=k$.

T.32 : $m \rceil\!\lfloor (k+k) \equiv m \rceil\!\lfloor k \;\&\; m \rceil\!\lfloor k$ ________ (1)

(1), P : $(Am)(m \rceil\!\lfloor (k+k) \equiv m \rceil\!\lfloor k)$ ________ (2)

(2), T.22 : $k+k=k$.

34. $k+l=l+k$.

T.32, P : $(Am)(m \rceil\!\lfloor (k+l) \equiv m \rceil\!\lfloor (l+k))$ _______ (1)

(1), T.22 : $k+l=l+k$.

35. $k+(l+m)=(k+l)+m$.

T.32 : $n \rceil\!\lfloor (k+(l+m)) \equiv n \rceil\!\lfloor k \;\&\; n \rceil\!\lfloor (l+m)$ ________ (1)

$(1),\ T.32,\ P : n\rceil\!\lfloor(k+(l+m)) \equiv n\rceil\!\lfloor k \ \&\ n\rceil\!\lfloor l \ \&\ n\rceil\!\lfloor m$ $\rule{2cm}{0.4pt}$ (2)

$T.32,\ P : n\rceil\!\lfloor((k+l)+m) \equiv n\rceil\!\lfloor k \ \&\ n\rceil\!\lfloor l \ \&\ n\rceil\!\lfloor m$ $\rule{2cm}{0.4pt}$ (3)

$(2),\ (3),\ P : (An)(n\rceil\!\lfloor(k+(l+m)) \equiv n\rceil\!\lfloor((k+l)+n))$ $\rule{2cm}{0.4pt}$ (4)

$(4),\ T.22 : k+(l+m) = (k+l)+m.$

36. $\underline{mo(k+l) \equiv mok \text{ v } mol.}$

$Defn.4,\ T.32,\ P : \sim(mo(k+l)) \equiv \sim(mok) \ \&\ \sim(mol)$ $\rule{2cm}{0.4pt}$ (1)

$(1),\ P : mo(k+l) \equiv mok \text{ v } mol.$

37. $\underline{(k+l) \leq m \equiv k \leq m \ \&\ l \leq m.}$

$Defn.1,\ P : (k+l) \leq m \equiv (An)(no(k+l) \supset nom)$ $\rule{2cm}{0.4pt}$ (1)

$(1),\ T.36,\ P : (k+l) \leq m \equiv (An)(nok \text{ v } nol \supset nom)$ $\rule{2cm}{0.4pt}$ (2)

$(2),\ P : (k+l) \leq m \equiv (An)(nok \supset nom) \ \&\ (An)(nol \supset nom)$ $\rule{2cm}{0.4pt}$ (3)

$(3),\ Defn.1,\ P : (k+l) \leq m \equiv k \leq m \ \&\ l \leq m.$

38. $\underline{k \leq k+l.}$

$T.32,\ P : m\rceil\!\lfloor(k+l) \supset m\rceil\!\lfloor k$ $\rule{2cm}{0.4pt}$ (1)

$(1),\ T.21,\ P : k \leq k+l.$

39. $\underline{(Sk)(\emptyset(k) \ \&\ l \leq k) \supset l \leq Fu'\{k : \emptyset(k)\}.}$

$Hyp : (Sk)(\emptyset(k) \ \&\ l \leq k)$ $\rule{2cm}{0.4pt}$ (1)

$T.31,\ (1),\ P : (Sk)(k \in \{k : \emptyset(k)\})$ $\rule{2cm}{0.4pt}$ (2)

$(1),\ P : \emptyset(k_1) \ \&\ l \leq k_1$ $\rule{2cm}{0.4pt}$ (3) (k_1 is a constant.)

$(2),\ T.27,\ P : m\rceil\!\lfloor Fu'\{k : \emptyset(k)\} \equiv (An)(n \in \{k : \emptyset(k)\} \supset m\rceil\!\lfloor n)$ $\rule{2cm}{0.4pt}$ (4)

$(3),\ T.21 : m\rceil\!\lfloor k_1 \supset m\rceil\!\lfloor l$ $\rule{2cm}{0.4pt}$ (5)

$(3),\ T.31,\ P : (An)(n \in \{k : \emptyset(k)\} \supset m\rceil\!\lfloor n) \supset m\rceil\!\lfloor k_1$ $\rule{2cm}{0.4pt}$ (6)

$(4),\ (5),\ (6),\ P : m\rceil\!\lfloor Fu'\{k : \emptyset(k)\} \supset m\rceil\!\lfloor l$ $\rule{2cm}{0.4pt}$ (7)

$(7),\ T.21 : l \leq Fu'\{k : \emptyset(k)\}$ $\rule{2cm}{0.4pt}$ (8)

(1), (8), P : $(Sk)(\phi(k) \,\&\, 1 \leq k) \supset 1 \leq Fu'\{k : \phi(k)\}$.

40. <u>$(Sk)\phi(k) \supset. (Ak)(\phi(k) \supset k \leq 1) \equiv Fu''\{k : \phi(k)\} \leq 1$.</u>

Hyp : $(Sk)\phi(k)$ ______(1)

(1), T.27, T.31 : $m \rceil\!\lfloor Fu'\{k : \phi(k)\} \equiv (An)(\phi(n) \supset m \rceil\!\lfloor n)$ ______(2)

Hyp : $m \rceil\!\lfloor 1 \supset m \rceil\!\lfloor Fu'\{k : \phi(k)\}$ ______(3)

(2), (3), P : $(An)(m \rceil\!\lfloor 1 \supset. \phi(n) \supset m \rceil\!\lfloor n)$ ______(4)

(4), T.21, P : $(Ak)(\phi(k) \supset k \leq 1)$ ______(5)

T.21, (3), (5), P : $Fu'\{k : \phi(k)\} \leq 1 \supset (Ak)(\phi(k) \supset k \leq 1)$ ______(6)

Hyp : $\phi(k) \supset k \leq 1$ ______(7)

Defn.1, (7), P : $(Am)(\phi(k) \supset. m \rceil\!\lfloor 1 \supset m \rceil\!\lfloor k)$ ______(8)

(8), P : $m \rceil\!\lfloor 1 \supset (Ak)(\phi(k) \supset m \rceil\!\lfloor k)$ ______(9)

(2), (9) : $m \rceil\!\lfloor 1 \supset m \rceil\!\lfloor Fu'\{k : \phi(k)\}$ ______(10)

(10), T.21 : $Fu'\{k : \phi(k)\} \leq 1$ ______(11)

(2), (11), P : $(Ak)(\phi(k) \supset k \leq 1) \supset Fu'\{k : \phi(k)\} \leq 1$ ______(12)

(6), (12) : $(Ak)(\phi(k) \supset k \leq 1) \equiv Fu'\{k : \phi(k)\} \leq 1$ ______(13)

(1), (13), P : T.40.

41. <u>$(Sk)\ \phi(k) \supset. (Ak)(\phi(k) \supset k \rceil\!\lfloor 1) \equiv Fu'\{k : \phi(k)\} \rceil\!\lfloor 1$.</u>

Hyp : $(Sk)\phi(k)$ ______(1)

(1), T.27, T.31 : $m \rceil\!\lfloor Fu'\{k : \phi(k)\} \equiv (An)(\phi(n) \supset m \rceil\!\lfloor n)$ ______(2)

(1), (2), P : T.41.

42. <u>$(Sk)\phi(k) \supset. (Sk)(\phi(k) \,\&\, k o 1) \equiv Fu'\{k : \phi(k)\}\, o 1$.</u>

Hyp : $(Sk)\phi(k)$ ______(1)

(1), T.41, P : $(Sk)(\phi(k) \,\&\, \sim k \rceil\!\lfloor 1) \equiv \sim Fu'\{k : \phi(k)\} \rceil\!\lfloor 1$ ______(2)

(1), (2), Defn.4, P : T.42.

Let us introduce the symbol $\bar{k}$ for $Fu^{\prime}\{1 : 1 \rceil\!\lfloor k\}$, provided (S1) $(1 \rceil\!\lfloor k)$ holds.

43. $\underline{(S1)(1 \rceil\!\lfloor k) \supset . 1 \rceil\!\lfloor \bar{k} \equiv 1 \leq k.}$

Hyp : $(S1)(1 \rceil\!\lfloor k)$ ______(1)

(1), T.41, Defn.$\bar{k}$: $1 \rceil\!\lfloor \bar{k} \equiv (Am)(m \rceil\!\lfloor k \supset m \rceil\!\lfloor 1)$ ______ (2)

(2), T.21 : $1 \rceil\!\lfloor \bar{k} \equiv 1 \leq k$ ______(3)

(1), (3) : T.43.

44. $\underline{(S1)(1 \rceil\!\lfloor k) \supset k \rceil\!\lfloor \bar{k}.}$

T.43 : $(S1)(1 \rceil\!\lfloor k) \supset . k \rceil\!\lfloor \bar{k} \equiv k \leq k$ ______(1)

(1), T.3, P : T.44.

45. $\underline{(S1)(1 \rceil\!\lfloor k) \supset . 1 \leq \bar{k} \equiv 1 \rceil\!\lfloor k.}$

Hyp : $(S1)(1 \rceil\!\lfloor k)$ ______(1)

Hyp : $1 \leq \bar{k}$ ______(2)

(2), T.21 : $(Am)(m \rceil\!\lfloor \bar{k} \supset m \rceil\!\lfloor 1)$ ______(3)

(1), (3), P : $k \rceil\!\lfloor 1$ ______(4)

(4), T.20 : $1 \rceil\!\lfloor k$ ______(5)

(2), (5), P : $1 \leq \bar{k} \supset 1 \rceil\!\lfloor k$ ______(6)

Hyp : $1 \rceil\!\lfloor k$ ______(7)

(7), T.21, P : $m \leq k \supset m \rceil\!\lfloor 1$ ______(8)

(8), T.43 : $m \rceil\!\lfloor \bar{k} \supset m \rceil\!\lfloor 1$ ______(9)

(9), T.21 : $1 \leq \bar{k}$ ______(10)

(7), (10), P : $1 \rceil\!\lfloor k \supset 1 \leq \bar{k}$ ______(11)

(1), (6), (11), P : T.45.

46. $(S1)(1 \rceil\!\lfloor k) \supset \bar{\bar{k}}=k.$

Hyp : $(S1)(1 \rceil\!\lfloor k)$ ______ (1)

(1), T.45, P : $1 \leq \bar{\bar{k}} \equiv 1 \rceil\!\lfloor \bar{k}$ ______ (2)

(1), T.43, P : $1 \rceil\!\lfloor \bar{k} \equiv 1 \leq k$ ______ (3)

(2), (3), P : $1 \leq \bar{\bar{k}} \equiv 1 \leq k$ ______ (4)

(4), T.18, P : $\bar{\bar{k}}=k$ ______ (5)

(1), (5) : T.46.

Let us introduce the symbol U for $Fu'\{k : k=k\}$. This will always be defined because of T.7.

47. $k \leq U.$

T.7, T.27, T.31 : $m \rceil\!\lfloor U \equiv (An)(n=n \supset m \rceil\!\lfloor n)$ ______ (1)

(1), T.7, P : $m \rceil\!\lfloor U \equiv (An)(m \rceil\!\lfloor n)$ ______ (2)

(2), P : $m \rceil\!\lfloor U \supset m \rceil\!\lfloor k$ ______ (3)

(3), T.21 : $k \leq U$.

48. $k \circ U.$

T.47, T.5 : $k \circ U$.

49. $k+U=U.$

T.38 : $U \leq k+U$ ______ (1)

T.47, P : $k+U \leq U$ ______ (2)

(1), (2), T.1 : $k+U=U$.

50. $(Ak)(k \leq 1) \equiv 1=U.$

Hyp : $(Ak)(k \leq 1)$ ______ (1)

(1), P : $U \leq 1$ ______ (2)

(2), T.47, T.1 : $1=U$ ______ (3)

(1), (3) : $(Ak)(k \lesssim l) \supset l=U$ ______(4)

Hyp : $l=U$ ________(5)

(5), T.25 : $(Ak)(k \lesssim U) \equiv (Ak)(k \lesssim l)$ ________(6)

(6), T.47 : $(Ak)(k \lesssim l)$ ______(7)

(5), (7) : $l=U \supset (Ak)(k \lesssim l)$ _______(8)

(4), (8) : T.50.

51. $(Ak)(kol) \equiv l=U.$

T.50, T.5, P : $l=U \supset (Ak)(kol)$ ______(1)

Hyp : $(Ak)(kol)$ _______(2)

(2), P : $(Am)(mok \supset mol)$ _______(3)

(3), Defn.1, P : $(Ak)(k \lesssim l)$ _______(4)

(4), T.50 : $l=U$ ________(5)

(2), (5), P : $(Ak)(kol) \supset l=U$ ________(6)

(1), (6) : T.51.

52. $(Sl)(l \rceil\!\lfloor k) \supset . k+\bar{k}=U.$

Hyp : $(Sl)(l \rceil\!\lfloor k)$ _______(1)

T.32 : $m \rceil\!\lfloor (k+\bar{k}) \equiv m \rceil\!\lfloor k \;\&\; m \rceil\!\lfloor \bar{k}$ ________(2)

(1), T.43 : $m \rceil\!\lfloor \bar{k} \equiv m \lesssim k$ _______(3)

T.5, Defn.4, P : $m \lesssim k \supset \sim(m \rceil\!\lfloor k)$ _______(4)

(2), (3), (4), P : $\sim(m \rceil\!\lfloor (k+\bar{k}))$ _________(5)

(5), Defn.4, P : $(Am)(mo(k+\bar{k}))$ ________(6)

(6), T.51 : $k+\bar{k}=U$_______(7)

(1), (7), P : T.52.

53. $\underline{(S1)(1 \rceil\!\lfloor k) \equiv {\sim}k{=}U.}$

T.51, Defn.4 : $(A1){\sim}(1 \rceil\!\lfloor k) \equiv k{=}U$ ______ (1)

(1), P : $(S1)(1 \rceil\!\lfloor k) \equiv {\sim}k{=}U.$

54. $\underline{{\sim}k{=}U \supset {\sim}\bar{k}{=}U.}$

Hyp : ${\sim}k{=}U$ _____ (1)

Hyp : $\bar{k}{=}U$ ______ (2)

(2), T.1 : $U \leqq \bar{k}$ ______ (3)

(1), T.53, (3), T.45 : $U \rceil\!\lfloor k$ ______ (4)

T.48, Defn.4 : ${\sim}(U \rceil\!\lfloor k)$ ______ (5)

(2), (4), (5), P : ${\sim}\bar{k}{=}U$ ______ (6)

(1), (6), P : T.54.

55. $\underline{(Sk)({\sim}k{=}U \;\&\; \phi(k)) \equiv (Sk)({\sim}k{=}U \;\&\; \phi(\bar{k})).}$

Hyp : $(Sk)({\sim}k{=}U \;\&\; \phi(k))$ ______ (1)

(1), P : ${\sim}k_1{=}U \;\&\; \phi(k_1)$ ______ (2) (k_1 is a constant.)

(2), T.46, T.25 : $\phi(\bar{\bar{k}}_1)$ _______ (3)

(2), T.54 : ${\sim}\bar{k}_1{=}U$ _______ (4)

(3), (4), P : $(Sk)({\sim}k{=}U \;\&\; \phi(\bar{k}))$ ______ (5)

(1), (5) : $(Sk)({\sim}k{=}U \;\&\; \phi(k)) \supset (Sk)({\sim}k{=}U \;\&\; \phi(\bar{k}))$ _____ (6)

Hyp : $(Sk)({\sim}k{=}U \;\&\; \phi(\bar{k}))$ ______ (7)

(7), P : ${\sim}k_1{=}U \;\&\; \phi(\bar{k}_1)$ ______ (8)

(8), T.54 : ${\sim}\bar{k}_1{=}U \;\&\; \phi(\bar{k}_1)$ ______ (9)

(9), P : $(Sk)({\sim}k{=}U \;\&\; \phi(k))$ _______ (10)

(7), (10), P : $(Sk)({\sim}k{=}U \;\&\; \phi(\bar{k})) \supset (Sk)({\sim}k{=}U \;\&\; \phi(k))$ ______ (11)

(6), (11) : T.55.

56. $(Ak)(\sim k{=}U \supset \emptyset(k)) \equiv (Ak)(\sim k{=}U \supset \emptyset(\bar{k}))$.

T.55, P : $\sim(Sk)(\sim k{=}U \ \&\ \sim\emptyset(k)) \equiv \sim(Sk)(\sim k{=}U \ \&\ \sim\emptyset(\bar{k}))$ ______(1)

(1), P : $(Ak)(\sim k{=}U \supset \emptyset(k)) \equiv (Ak)(\sim k{=}U \supset \emptyset(\bar{k}))$.

57. $\sim k{=}U \ \&\ \sim l{=}U \supset . k{=}l \equiv \bar{k}{=}\bar{l}$.

Hyp : $\sim k{=}U \ \&\ \sim l{=}U$ ______(1)

T.22 : $k{=}l \equiv (Am)(m \rceil k \equiv m \rceil l)$ ________(2)

(1), (2), T.45 : $k{=}l \equiv (Am)(m \leq \bar{k} \equiv m \leq \bar{l})$ _______(3)

(3), T.18 : $k{=}l \equiv \bar{k}{=}\bar{l}$ _______(4)

(1), (4) : T.57.

58. $\sim k{=}U \ \&\ \sim l{=}U \supset . k \leq l \equiv \bar{l} \leq \bar{k}$.

Hyp : $\sim k{=}U \ \&\ \sim l{=}U$ _______(1)

T.21 : $k \leq l \equiv (Am)(m \rceil l \supset m \rceil k)$ ________(2)

(1), (2), T.45 : $k \leq l \equiv (Am)(m \leq \bar{l} \supset m \leq \bar{k})$ ________(3)

(3), T.17 : $k \leq l \equiv \bar{l} \leq \bar{k}$ _______(4)

(1), (4) : T.58.

We now introduce the definition of nucleus of a set as follows :

(I) $Nu^{\backprime}f' =_{df} \overline{\overline{Fu}}\{k : (Sl)(l \in f' \ \&\ \sim l{=}U \ \&\ k{=}\bar{l})\}$, provided (i) $(Sk)(Sl)(l \in f' \ \&\ \sim l{=}U \ \&\ k{=}\bar{l})$ and (ii) $\sim Fu^{\backprime}\{k : (Sl)(l \in f' \ \&\ \sim l{=}U \ \&\ k{=}\bar{l})\} {=} U$ both hold. These conditions are necessary to ensure the definability of (i) $Fu^{\backprime}\{k : (Sl)(l \in f' \ \&\ \sim l{=}U \ \&\ k{=}\bar{l})\}$ and (ii) $\overline{Fu}^{\backprime}\{\ldots .\}$. However this definition does not cover the case where U is the only member of f'. So a special definition is given in this case.

(II) $Nu^{\backprime}f' =_{df} U$, provided $(Ak)(k \in f' \supset k{=}U)$ holds.

59. $\underline{(Sk)(Sl)(l\in f' \;\&\; \sim l=U \;\&\; k=\bar{l}) \equiv (Sk)(k\in f' \;\&\; \sim k=U).}$

Hyp : $(Sk)(Sl)(l\in f' \;\&\; \sim l=U \;\&\; k=\bar{l})$ ______(1)

(1), P : $l_1\in f' \;\&\; \sim l_1=U \;\&\; k_1=\bar{l}_1$ ______(2) (k_1 and l_1 are constants.)

(2), P : $l_1\in f' \;\&\; \sim l_1=U$ ______(3)

(3), P : $(Sk)(k\in f' \;\&\; \sim k=U)$ ______(4)

(1), (4) : $(Sk)(Sl)(l\in f' \;\&\; \sim l=U \;\&\; k=\bar{l}) \supset (Sk)(k\in f' \;\&\; \sim k=U)$ ______(5)

Hyp : $(Sk)(k\in f' \;\&\; \sim k=U)$ ______(6)

(6), P : $k_1\in f' \;\&\; \sim k_1=U$ ______(7) (k_1 is a constant.)

T.57, (7) : $\bar{k}_1=\bar{k}_1$ ______(8)

(7), (8), P : $(Sl)(l\in f' \;\&\; \sim l=U \;\&\; \bar{k}_1=\bar{l})$ ______(9)

(9), P : $(Sk)(Sl)(l\in f' \;\&\; \sim l=U \;\&\; k=\bar{l})$ ______(10)

(6), (10) : $(Sk)(k\in f' \;\&\; \sim k=U) \supset (Sk)(Sl)(l\in f' \;\&\; \sim l=U \;\&\; k=\bar{l})$ ______(11)

(5), (11) : T.59.

60. $\underline{(Sk)(k\in f' \;\&\; \sim k=U) \supset. \sim Fu'\{k : (Sl)(l\in f' \;\&\; \sim l=U \;\&\; k=\bar{l})\}=U \equiv}$

$\underline{(Sl)(Ak)(k\in f' \supset l \leq k).}$

Hyp : $(Sk)(k\in f' \;\&\; \sim k=U)$ ______(1)

(1), T.53 : $\sim Fu'\{\ldots.\}=U \equiv (Sm)(m \rceil\!\lfloor Fu'\{\ldots.\})$ ______(2)

(1), (2), T.27 : $\sim Fu'\{\ldots.\}=U \equiv (Sm)(Ak)((Sl)(l\in f' \;\&\; \sim l=U \;\&\; k=\bar{l})$
$\supset m \rceil\!\lfloor k)$ ______(3)

(3), P : $\sim Fu'\{\ldots.\}=U \equiv (Sm)(Ak)((Al)(\sim l\in f' \vee l=U \vee \sim k=\bar{l}) \vee m \rceil\!\lfloor k)$
______(4)

(4), P : $\sim Fu'\{\ldots.\}=U \equiv (Sm)(Al)(l\in f' \;\&\; \sim l=U \supset (Ak)(k=\bar{l} \supset m \rceil\!\lfloor k))$
______(5)

P : $(Ak)(k=\bar{l} \supset m \rceil\!\lfloor k) \supset m \rceil\!\lfloor \bar{l}$ ______(6)

T.22, P : $m \rceil \bar{l} \supset (Ak)(k=\bar{l} \supset m \rceil k)$ ______(7)

(5), (6), (7), T.43, P : $\sim Fu'\{\ldots\}=U \equiv (Sm)(Al)(l \in f' \ \& \sim l=U \supset m \leq l)$ ______(8)

T.47 : $l=U \supset m \leq l$ ______ (9)

(8), (9), P : $\sim Fu'\{\ldots\}=U \equiv (Sm) \quad (Al)(l \in f' \supset m \leq l)$ ______(10)

(1), (10), P : T.60.

T.59 and T.60 simplify the conditions on the definition of $Nu'f'$ so that condition (i) becomes $(Sk)(k \in f' \ \& \sim k=U)$ and condition (ii) becomes $(Sl)(Ak)(k \in f' \supset l \leq k)$.

61. <u>$(Sl)(Ak)(k \in f' \supset l \leq k) \supset (Am)(m \leq Nu'f' \equiv (Al)(l \in f' \supset m \leq l))$.</u>

Hyp : $(Sl)(Ak)(k \in f' \supset l \leq k)$ ______(1)

Hyp : $(Sk)(k \in f' \ \& \sim k=U)$ ______(2)

(2), T.27 : $m \rceil Fu'\{k : (Sl)(l \in f' \ \& \sim l=U \ \& \ k=\bar{l})\} \equiv (Ak)((Sl)(l \in f' \ \& \sim l=U \ \& \ k=\bar{l}) \supset m \rceil k)$ ______ (3)

(3), Defn.Nu, T.45, (1) : $m \leq Nu'f' \equiv (Ak)((Sl)(l \in f' \ \& \sim l=U \ \& \ k=\bar{l}) \supset m \rceil k)$ ______(4)

Using the proof of T.60, $m \leq Nu'f' \equiv (Al)(l \in f' \supset m \leq l)$ ______(5)

(2), (5) : $(Sk)(k \in f' \ \& \sim k=U) \supset (Am)(m \leq Nu'f' \equiv (Al)(l \in f' \supset m \leq l))$ ______(6)

Hyp : $(Ak)(k \in f' \supset k=U)$ ______(7)

T.47 : $k=U \supset m \nleq k$ ______(8)

(7), (8) : $(Al)(l \in f' \supset m \leq l)$ ______(9)

T.47, Defn.Nu : $m \leq Nu'f'$ ______(10)

(9), (10), P : $(Am)(m \leq Nu'f' \equiv (Al)(l \in f' \supset m \leq l))$ ______(11)

$(7), (11) : (Ak)(k \in f' \supset k=U) \supset (Am)(m \leq Nu‘f' \equiv (Al)(l \in f' \supset m \leq l))$ ______(12)

$(1), (6), (12), P : T.61.$

62. <u>$(Sl)(Ak)(k \in f' \supset l \leq k) \supset . kNuf' \equiv k=Nu‘f'.$</u>

$T.61, Defn.6, T.25, T.18 : T.62.$

63. <u>$(Sl)(Ak)(k \in g' \supset l \leq k) \supset . f' \subseteq g' \supset Nu‘g' \leq Nu‘f'.$</u>

Hyp : $(Sl)(Ak)(k \in g' \supset l \leq k)$ ______(1)

Hyp : $f' \subseteq g'$ ______(2)

$(1), (2), Defn.7, P : (Sl)(Ak)(k \in f' \supset l \leq k)$ ______(3)

$(1), T.61 : m \leq Nu‘f' \equiv (Al)(l \in f' \supset m \leq l)$ ______(4)

$(1), T.61 : m \leq Nu‘g' \equiv (Al)(l \in g' \supset m \leq l)$ ______(5)

$(2), Defn.7, P : (Al)(l \in g' \supset m \leq l) \supset (Al)(l \in f' \supset m \leq l)$ ______(6)

$(4), (5), (6), P : m \leq Nu‘g' \supset m \leq Nu‘f'$ ______(7)

$(7), T.17 : Nu‘g' \leq Nu‘f'$ ______(8)

$(1), (2), (8) : T.63.$

64. <u>$(Sl)(Ak)(k \in f' \supset l \leq k) \supset . f'=g' \supset Nu‘f'=Nu‘g'.$</u>

Hyp : $(Sl)(Ak)(k \in f' \supset l \leq k)$ ______(1)

Hyp : $f'=g'$ ______(2)

(2), Defns.7 and 8 : $f' \subseteq g' \ \& \ g' \subseteq f'$ ______(3)

$(1), (2), T.25 : (Sl)(Ak)(k \in g' \supset l \leq k)$ ______(4)

$(1), (3), (4), T.63 : Nu‘f' \leq Nu‘g' \ \& \ Nu‘g' \leq Nu‘f'$ ______(5)

$(5), T.1 : Nu‘f'=Nu‘g'$ ______(6)

$(1), (2), (6) : T.64.$

65. $\underline{(Sl)(Ak)(\emptyset(k) \supset l \leq k) \supset. (Ak)(\emptyset(k) \supset l \leq k) \equiv l \leq Nu'\{k : \emptyset(k)\}}$.

T.61, T.31 : T.65.

66. $\underline{(Sl)(Ak)(\emptyset(k) \supset l \leq k) \supset. (Sk)(\emptyset(k) \;\&\; k \leq l) \supset Nu'\{k : \emptyset(k)\} \leq l}$.

Hyp : $(Sl)(Ak)(\emptyset(k) \supset l \leq k)$ _____ (1)

Hyp : $(Sk)(\emptyset(k) \;\&\; k \leq l)$ _____ (2)

(2), P : $\emptyset(k_1) \;\&\; k_1 \leq l$ _____ (3) (k_1 is a constant.)

(1), T.31, T.61 : $(Am)(m \leq Nu'\{k : \emptyset(k)\} \supset (Al)(\emptyset(l) \supset m \leq l))$ _____ (4)

(3), (4) : $(Am)(m \leq Nu'\{k : \emptyset(k)\} \supset m \leq k_1)$ _____ (5)

(5), T.17 : $Nu'\{k : \emptyset(k)\} \leq k_1$ _____ (6)

(3), (6), T.10 : $Nu'\{k : \emptyset(k)\} \leq l$ _____ (7)

(1), (2), (7) : T.66.

67. $\underline{(Sl)(Ak)(\emptyset(k) \supset l \leq k) \supset. (Sk)(\emptyset(k) \;\&\; k \rceil l) \supset Nu'\{k : \emptyset(k)\} \rceil l}$.

Hyp : $(Sl)(Ak)(\emptyset(k) \supset l \leq k)$ _____ (1)

Hyp : $\sim l = U$ _____ (2)

(2), T.43, T.66 : $(Sk)(\emptyset(k) \;\&\; k \rceil \bar{l}) \supset Nu'\{k : \emptyset(k)\} \rceil \bar{l}$ _____ (3)

(2), (3), T.56 : $(Sk)(\emptyset(k) \;\&\; k \rceil l) \supset Nu'\{k : \emptyset(k)\} \rceil l$ _____ (4)

(2), (4) : $\sim l = U \supset. (Sk)(\emptyset(k) \;\&\; k \rceil l) \supset Nu'\{k : \emptyset(k)\} \rceil l$ _____ (5)

Hyp : $l = U$ _____ (6)

T.48, Defn.4, P : $(Sk)(\emptyset(k) \;\&\; k \rceil U) \supset Nu'\{k : \emptyset(k)\} \rceil U$ _____ (7)

(6), (7), T.25 : $(Sk)(\emptyset(k) \;\&\; k \rceil l) \supset Nu'\{k : \emptyset(k)\} \rceil l$ _____ (8)

(6), (8) ; $l = U \supset. (Sk)(\emptyset(k) \;\&\; k \rceil l) \supset Nu'\{k : \emptyset(k)\} \rceil l$ _____ (9)

(1), (5), (9) : T.67.

68. $\underline{(Sl)(Ak)(\emptyset(k) \supset l \leq k) \supset. l \, o \, Nu'\{k : \emptyset(k)\} \supset (Ak)(\emptyset(k) \supset l \, o \, k)}$.

Defn.4, T.67, P : T.68.

We now introduce the definition of kl as $Nu`\{k,l\}$, subject to the condition that kol holds. The condition is correct for this nucleus because :

$(Sn)(Am)(m\in\{k,l\} \supset n\leq m)$

$\equiv (Sn)(Am)(m=k \vee m=l \supset n\leq m)$ (Defn. k,l)

$\equiv (Sn)(Am)(m=k \supset n\leq m . \& . m=l \supset n\leq m)$

$\equiv (Sn)((Am)(m=k \supset n\leq m) \; \& \; (Am)(m=l \supset n\leq m))$

$\equiv (Sn)(n\leq k \; \& \; n \leq l)$ (T.7, T.18)

$\equiv$ kol (T.2)

69. <u>$kol \supset . m\leq k \; \& \; m\leq l \equiv m\leq kl.$</u>

Hyp : kol _____(1)

(1), T.61, Defn.kl : $m\leq kl \equiv (An)(n\in\{k,l\} \supset m \leq n)$ _____(2)

(2), Defn. k,l : $m\leq kl \equiv (An)(n=k \vee n=l \supset m\leq n)$ _______(3)

(3), P : $m\leq kl \equiv (An)(n=k \supset m\leq n) \; \& \; (An)(n=l \supset m\leq n)$ ______(4)

(4), T.7, T.18 : $m\leq kl \equiv m\leq k \; \& \; m\leq l$______(5)

(1), (5) : T.69.

70. <u>$kol \supset kl\leq k.$</u>

T.69 : $kol \supset . m\leq kl \supset m\leq k$ ______(1)

(1), T.17 : $kol \supset kl\leq k.$

71. <u>$kol \supset . k \urcorner m \supset kl \urcorner m.$</u>

T.70, T.21 : $kol \supset (Am)(m \urcorner k \supset m \urcorner kl)$ _____(1)

(1), T.20, P : T.71.

72. <u>$kol \supset . mokl \supset mok \; \& \; mol.$</u>

T.69 : $kol \supset . m \leq kl \supset m \leq l$ _____(1)

(1), T.17 : $kol \supset . kl \leqslant l$ ______(2)

(2), Defn.1 : $kol \supset . mokl \supset mol$ ______(3)

T.70, Defn.1 : $kol \supset . mokl \supset mok$ ______(4)

(3), (4) : $kol \supset . mokl \supset mok \;\&\; mol$.

73. <u>$kk=k$.</u>

T.4, T.69 : $m \leqslant k \;\&\; m \leqslant k \equiv m \leqslant kk$ ______(1)

(1), P : $(Am)(m \leqslant k \equiv m \leqslant kk)$ ______(2)

(2), T.18 : T.73.

74. <u>$kol \supset kl=lk$.</u>

Hyp : kol ______(1)

(1), T.69 : $m \leqslant kl \equiv m \leqslant lk$ ______(2)

(2), T.18 : $kl=lk$ ______(3)

(1), (3) : T.74.

75. <u>$(Sn)(n \leqslant k \;\&\; n \leqslant l \;\&\; n \leqslant m) \supset . k(lm)=(kl)m$.</u>

Hyp : $(Sn)(n \leqslant k \;\&\; n \leqslant l \;\&\; n \leqslant m)$ ______(1)

(1), T.69 : $(Sn)(n \leqslant k \;\&\; n \leqslant lm)$ ______(2)

(1), T.69 : $(Sn)(n \leqslant kl \;\&\; n \leqslant m)$ ______(3)

(1), T.2 : kol ______(4)

(1), T.2 : lom ______(5)

(2), T.2 : $ko(lm)$ ______(6)

(3), T.2 : $(kl)om$ ______(7)

(6), T.69 : $n \leqslant k(lm) \equiv n \leqslant k \;\&\; n \leqslant lm$ ______(8)

(5), (8), T.69 : $n \leqslant k(lm) \equiv n \leqslant k \;\&\; n \leqslant l \;\&\; n \leqslant m$ ______(9)

(7), T.69 : $n \leqslant (kl)m \equiv n \leqslant kl \;\&\; n \leqslant m$ ______(10)

(4), (10), T.69 : $n \lesssim (kl)m \equiv n \lesssim k \,\&\, n \lesssim l \,\&\, n \lesssim m$ ______(11)

(9), (11) : $n \lesssim k(lm) \equiv n \lesssim (kl)m$ ______(12)

(12), T.18 : $k(lm)=(kl)m$ ______(13)

(1), (13) : T.75.

76. <u>$kU=k$.</u>

T.48 : koU ______(1)

(1), T.69 : $m \lesssim kU \equiv m \lesssim k \,\&\, m \lesssim U$ ______(2)

(2), T.47 : $m \lesssim kU \equiv m \lesssim k$ ______(3)

(3), T.18 : $kU=k$.

77. <u>$kol \,\&\, \sim k=U \,\&\, \sim l=U \supset . \overline{kl}=\bar{k}+\bar{l}$.</u>

Hyp : $kol \,\&\, \sim k=U \,\&\, \sim l=U$ ______(1)

Hyp : $kl=U$ ______(2)

(2), T.18 : $m \lesssim kl \equiv m \lesssim U$ ______(3)

(3), T.47 : $m \lesssim kl$ ______(4)

(1), (4), T.69 : $m \lesssim k \,\&\, m \lesssim l$ ______(5)

(5), T.50 : $k=U \,\&\, l=U$ ______(6)

(1), (2), (6) : $\sim kl=U$ ______(7)

T.69 : $m \lesssim kl \equiv m \lesssim k \,\&\, m \lesssim l$ ______(8)

(8), T.43 : $m \rceil\!\lfloor \overline{kl} \equiv m \rceil\!\lfloor \bar{k} \,\&\, m \rceil\!\lfloor \bar{l}$ ______(9)

(9), T.32 : $m \rceil\!\lfloor \overline{kl} \equiv m \rceil\!\lfloor (\bar{k}\ \bar{l})$ ______(10)

(10), T.22 : $\overline{kl}=\bar{k}+\bar{l}$ ______(11)

(1), (11) : T.77.

78. <u>$k \lesssim l \equiv k+l=l$.</u>

T.10 : $k \lesssim l \equiv (Am)(l \lesssim m \supset k \lesssim m)$ ______(1)

(1), P : $k \lesssim l \equiv (Am)(k \lesssim m \ \& \ l \lesssim m \equiv l \lesssim m)$ ______(2)

(2), T.37 : $k \lesssim l \equiv (Am)((k+l) \lesssim m \equiv l \lesssim m)$ ______(3)

(3), P : $k \lesssim l \equiv (Am)(\sim m = U \supset . k+l \lesssim m \equiv l \lesssim m) \ \& \ (Am)(m = U \supset . \ k+l \lesssim m$
$\equiv l \lesssim m)$ ______(4)

(4), T.47, T.25 : $k \lesssim l \equiv (Am)(\sim m = U \supset . k+l \lesssim m \equiv l \lesssim m)$ ______(5)

(5), T.56, T.45 : $k \lesssim l \equiv (Am)(\sim m = U \supset . k+l \rceil\!\lfloor m \equiv l \rceil\!\lfloor m)$ ______(6)

(6), T.48, T.25 : $k \lesssim l \equiv (Am)(m \rceil\!\lfloor (k+l) \equiv m \rceil\!\lfloor l)$ ______(7)

(7), T.22 : $k \lesssim l \equiv k+l = l$.

79. <u>$k \lesssim l \ \& \ m \lesssim n \supset k+m \lesssim l+n$.</u>

T.78 : $k \lesssim l \ \& \ m \lesssim n \supset k+l = l \ \& \ m+n = n$ ______(1)

(1), T.34, T.35 : $k \lesssim l \ \& \ m \lesssim n \supset l+n = (k+m)+(l+n)$ ______(2)

(3), T.78 : $k \lesssim l \ \& \ m \lesssim n \supset k+m \lesssim l+n$.

80. <u>$kol \supset . k \lesssim l \equiv kl = k$.</u>

Hyp : kol ______(1)

(1), T.69 : $k \lesssim l \equiv k \lesssim kl$ ______(2)

(1), T.70 : $kl \lesssim k$ ______(3)

(2), (3) : $k \lesssim l \equiv k \lesssim kl \ \& \ kl \lesssim k$ ______(4)

(4), T.1 : $k \lesssim l \equiv kl = k$ ______(5)

(1), (5) : T.80.

81. <u>$kom \supset . k \lesssim l \ \& \ m \lesssim n \supset km \lesssim ln$.</u>

Hyp : kom ______(1)

Hyp : $k \lesssim l \ \& \ m \lesssim n$ ______(2)

(1), (2), Defn.1 : lom ______(3)

(2), (3), Defn.1 : lon ______(4)

T.80 : $k \leq l$ & $m \leq n \supset kl=k$ & $mn=m$ ____(5)

(5), T.74, T.75 : $k \leq l$ & $m \leq n \supset km=(km)(ln)$ ____(6)

(6), T.80 : $k \leq l$ & $m \leq n \supset km \leq ln$.

82. <u>kol & $kom \supset kl+km = k(l+m)$.</u>

Hyp : kol & kom ______(1)

(1), T.38, Defn.1 : $ko(l+m)$ _____(2)

T.38 : $l \leq l+m$ & $m \leq l+m$ ______(3)

(1), T.81, T.3 : $kl \leq k(l+m)$ & $km \leq k(l+m)$ _____(4)

(4), T.37 : $kl+km \leq k(l+m)$ ______(5)

Hyp : $nok(l+m)$ ____(6)

(6), T.2 : $n_1 \leq n$ & $n_1 \leq k(l+m)$ _____(7) (n_1 is a constant.)

(7), T.69 : $n_1 \leq k$ & $n_1 \leq l+m$ ______(8)

(8), T.5 : $n_1 o(l+m)$ ______(9)

(9), T.36 : $n_1 ol$ v $n_1 om$ ______(10)

Hyp : $n_1 ol$ ______(11)

(11), T.2 : $n_2 \leq n_1$ & $n_2 \leq l$ _____(12) (n_2 is a constant.)

(8), (12), T.10 : $n_2 \leq k$ & $n_2 \leq l$ ______(13)

(13), T.69 : $n_2 \leq kl$ _____(14)

(7), (12), T.10 : $n_2 \leq n$ ______(15)

(14), (15), T.2 : $no(kl)$ ______(16)

(11), (16) : $n_1 ol \supset no(kl)$ _______(17)

Hyp : $n_1 om$ _____(18)

Similarly, $no(km)$ ______(19)

(18), (19) : $n_1 om \supset no(km)$ _____(20)

(10), (17), (20) : $no(kl) \vee no(km)$ _____(21)

(21), T.36 : $no(kl+km)$ _____(22)

(6), (22) : $(An)(nok(l+m) \supset no(kl+km))$ _____(23)

(23), Defn.1 : $k(l+m) \leq kl+km$ _____(24)

(1), (5), (24), T.1 : T.82.

83. <u>$lom \supset k+lm=(k+l)(k+m)$.</u>

Hyp : lom _____(1)

(1), T.36 : $(k+l)om$ _____(2)

(2), T.36 : $(k+l)o(k+m)$ _____(3)

T.38 : $k \leq k+l$ & $k \leq k+m$ _____(4)

(3), (4), T.69 : $k \leq (k+l)(k+m)$ _____(5)

T.38 : $l \leq k+l$ & $m \leq k+m$ _____(6)

(1), (6), T.81 : $lm \leq (k+l)(k+m)$ _____(7)

(7), T.37 : $(k+lm) \leq (k+l)(k+m)$ _____(8)

Hyp : $no(k+l)(k+m)$ _____(9)

(9), T.2, T.69 : $n_1 \leq n$ & $n_1 \leq k+l$ & $n_1 \leq k+m$ _____(10) (n_1 is a constant.)

Hyp : n_1ok _____(11)

(10), (11), Defn.1 : nok _____(12)

(11), (12) : $n_1ok \supset nok$ _____(13)

Hyp : $n_1 \mathrel{\underline{]}} k$ _____(14)

Hyp : n_2on_1 _____(15)

(15), T.2 : $n_3 \leq n_2$ & $n_3 \leq n_1$ _____(16) (n_3 is a constant.)

(14), (16), T.21 : $n_3 \mathrel{\underline{]}} k$ _____(17)

Hyp : $n_2 \rceil\!\lfloor 1$ ______ (18)

(16), (18), T.21 : $n_3 \rceil\!\lfloor 1$ ______ (19)

(17), (19), T.32 : $n_3 \rceil\!\lfloor (k+l)$ ______ (20)

(10), (20), T.21 : $n_3 \rceil\!\lfloor n_1$ ______ (21)

(16), T.5 : $n_3 o n_1$ ______ (22)

(21), (22), (15), (18) : $n_2 o n_1 \supset n_2 o l$ ______ (23)

(23), Defn.1 : $n_1 \leqslant l$ ______ (24)

Similarly, $n_1 \leqslant m$ ______ (25)

(24), (25), T.69 : $n_1 \leqslant lm$ ______ (26)

(10), (26), T.2 : $no(lm)$ ______ (27)

(14), (27) : $n_1 \rceil\!\lfloor k \supset no(lm)$ ______ (28)

(13), (28) : $nok \vee no(lm)$ ______ (29)

(29), T.36 : $no(k+lm)$ ______ (30)

(9), (30), Defn.1 : $(k+l)(k+m) \leqslant (k+lm)$ ______ (31)

(8), (31), T.1 : $k+lm = (k+l)(k+m)$ ______ (32)

(1), (32) : T.83.

84. <u>$kol \;\&\; \sim(k \leqslant l) \;\&\; \sim l = U \supset k = kl + k\bar{l}$.</u>

Hyp : $kol \;\&\; \sim(k \leqslant l) \;\&\; \sim l = U$ ______ (1)

(1), T.43 : $ko\bar{l}$ ______ (2)

(1), (2), T.82 : $kl + k\bar{l} = k(l+\bar{l})$ ______ (3)

(1), T.52 : $l+\bar{l} = U$ ______ (4)

(3), (4), T.25 : $kl+k\bar{l} = kU$ ______ (5)

(5), T.76 : $kl+k\bar{l} = k$ ______ (6)

(1), (6) : T.84.

85. $\underline{\sim 1 = U \supset k = (k+1)(k+\bar{1}).}$

Hyp : $\sim 1=U$ ______ (1)

T.38 : $k \leq k+1 \;\&\; k \leq k+\bar{1}$ ______ (2)

(2), T.2 : $(k+1)o(k+\bar{1})$ ______ (3)

(2), T.69 : $k \leq (k+1)(k+\bar{1})$ ______ (4)

Hyp : $no(k+1)(k+\bar{1})$ ______ (5)

(5), T.2, T.69 : $n_1 \leq n \;\&\; n_1 \leq k+1 \;\&\; n_1 \leq k+\bar{1}$ ______ (6) (n_1 is a constant.)

Hyp : $n_1 ok$ ______ (7)

(6), (7), Defn.1 : nok ______ (8)

(7), (8) : $n_1 ok \supset nok$ ______ (9)

Hyp : $n_1 \urcorner\!\llcorner k$ ______ (10)

Hyp : $n_2 on_1$ ______ (11) (n_2 is a constant.)

(11), T.2 : $n_3 \leq n_2 \;\&\; n_3 \leq n_1$ ______ (12) (n_3 is a constant.)

(10), (12), T.21 : $n_3 \urcorner\!\llcorner k$ ______ (13)

Hyp : $n_2 \urcorner\!\llcorner 1$ ______ (14)

(12), (14), T.21 : $n_3 \urcorner\!\llcorner 1$ ______ (15)

(13), (15), T.32 : $n_3 \urcorner\!\llcorner (k+1)$ ______ (16)

(6), (16), T.21 : $n_3 \urcorner\!\llcorner n_1$ ______ (17)

(12), T.5 : $n_3 on_1$ ______ (18)

(17), (18), (11), (14) : $n_2 on_1 \supset n_2 o1$ ______ (19)

(19), Defn.1 : $n_1 \leq 1$ ______ (20)

Similarly, $n_1 \leq \bar{1}$ ______ (21)

(21), T.45 : $n_1 \urcorner\!\llcorner 1$ ______ (22)

(20), (22), T.21 : $n_1 \rceil\!\lfloor n_1$ _____(23)

T.4 : $n_1 o n_1$ ____(24)

(10), (24) : $n_1 o k$ _____(25)

(25), (9) : $n o k$ _____(26)

(5), (26) : $n o (k+l)(k+\bar{l}) \supset n o k$ ____(27)

(27), Defn.1 : $(k+l)(k+\bar{l}) \leq k$ _____(28)

(4), (28), T.1 : $k = (k+l)(k+\bar{l})$ _____(29)

(1), (29) : T.85.

86. $\underline{\sim k=U \ \& \ \sim l=U \ \& \ k o l \ \& \ \sim(k \leq l) \ \& \ \sim(l \leq k) \supset k+l = kl+\bar{k}l+k\bar{l}.}$

Hyp : $\sim k=U \ \& \ \sim l=U \ \& \ k o l \ \& \ \sim(k \leq l) \ \& \ \sim(l \leq k)$ _____(1)

(1), T.43 : $\sim(k \rceil\!\lfloor \bar{l}) \ \& \ \sim(l \rceil\!\lfloor \bar{k})$ ______(2)

(2), Defn.4 : $k o \bar{l} \ \& \ l o \bar{k}$ _______(3)

(1), T.84 : $k+l = (kl+k\bar{l})+(lk+l\bar{k})$ _____(4)

(1), T.74 : $kl = lk$ ____(5)

(4), (5), T.25, T.33, T.34, T.35 : $k+l = kl+k\bar{l}+l\bar{k}$ ____(6)

(1), (6) : T.86.

87. $\underline{\sim k=U \ \& \ \sim l=U \ \& \ k o l \ \& \ \sim(k \leq l) \ \& \ \sim(l \leq k) \supset kl = (k+l)(k+\bar{l})(\bar{k}+l).}$

Hyp : $\sim k=U \ \& \ \sim l=U \ \& \ k o l \ \& \ \sim(k \leq l) \ \& \ \sim(l \leq k)$ ______(1)

(1), T.43, Defn.4 : $k o \bar{l} \ \& \ l o \bar{k}$ ______(2)

(1), T.85 : $kl = (k+l)(k+\bar{l})(l+k)(l+\bar{k})$ ______(3)

(1), (3), T.73, T.74, T.75, T.34 : $kl = (k+l)(k+\bar{l})(l+\bar{k})$ ____(4)

(1), (4) : T.87.

88. $\underline{f'=g' \supset . \emptyset(f') \equiv \emptyset(g')}$, for any wff $\emptyset$.

Defn.8 : $f'=g' \supset . k \in f' \equiv k \in g'$ ______(1)

Using (1), by induction on the numbers of connectives and quantifiers of $\emptyset$, T.88 can be shown.

89. $(S!h')(Ak)(k \in h' \equiv k \in f' \;\&\; k \in g')$.

A.3 : $(Sh')(Ak)(k \in h' \equiv k \in f' \;\&\; k \in g')$ ____(1)

Hyp : $(Ak)(k \in h_1' \equiv k \in f' \;\&\; k \in g') \;\&\; (Ak)(k \in h_2' \equiv k \in f' \;\&\; k \in g')$ ___(2)

(2), $(Ak)(k \in h_1' \equiv k \in h_2')$ ____(3)

(3), Defn.8 : $h_1' = h_2'$ _____(4)

(1), (2), (4) : T.89.

We now introduce the term $f' \cap g'$ for the unique h' such that $k \in h' \equiv k \in f' \;\&\; k \in g'$.

90. $(S!h')(Ak)(k \in h' \equiv k \in f' \vee k \in g')$.

A.3 : $(Sh')(Ak)(k \in h' \equiv k \in f' \vee k \in g')$ _____(1)

Hyp : $(Ak)(k \in h_1' \equiv k \in f' \vee k \in g') \;\&\; (Ak)(k \in h_2' \equiv k \in f' \vee k \in g')$ ___(2)

(2) : $(Ak)(k \in h_1' \equiv k \in h_2')$ ____(3)

(3), Defn.8 : $h_1' = h_2'$ _____(4)

(1), (2), (4) : T.90.

We now introduce the term $f' \cup g'$ for the unique h' such that $k \in h' \equiv k \in f' \vee k \in g'$.

91. $(S!h')(Ak)(k \in h' \equiv \sim k \in f')$.

A.3 : $(Sh')(Ak)(k \in h' \equiv \sim k \in f')$ _____(1)

Hyp : $(Ak)(k \in h_1' \equiv \sim k \in f') \;\&\; (Ak)(k \in h_2' \equiv \sim k \in f')$ ____(2)

(2) : $(Ak)(k \in h_1' \equiv k \in h_2')$ _____(3)

(3), Defn.8 : $h_1' = h_2'$ _____(4)

(1), (2), (4) : T.91.

We now introduce the term $\overline{\overline{f'}}$ for the unique h' such that k∈h' ≡ ~k∈f'.

92. <u>(Sk)(k∈f' & k∈g') ⊃ Fu‘(f'∩g') ≤ (Fu‘f')(Fu‘g').</u>

Hyp : (Sk)(k∈f' & k∈g') _____(1)

(1), Defn.∩ : (Sk)(k∈f'∩g') _____(2)

(1), T.39 : (Sk)(k ≤ Fu‘f' & k ≤ Fu‘g') _____(3)

(3), T.2 : (Fu‘f')o(Fu‘g') _____(4)

(2), T.41 : k⌉⌊Fu‘f'∩g' ≡ (Al)(l∈f' & l∈g' ⊃ l⌉⌊k) _____(5)

(1), T.41 : k⌉⌊Fu‘f' ≡ (Al)(l∈f' ⊃ l⌉⌊k) _____(6)

(5), (6) : k⌉⌊Fu‘f' ⊃ k⌉⌊Fu‘f'∩g' _____(7)

(7), T.21 : Fu‘f'∩g' ≤ Fu‘f' _____(8)

Similarly, Fu‘f'∩g' ≤ Fu‘g' _____(9)

(4), (8), (9), T.69 : Fu‘f'∩g' ≤ (Fu‘f')(Fu‘g') _____(10)

(1), (10) : T.92.

93. <u>(Sk)(k∈f') & (Sk)(k∈g') ⊃.Fu‘(f'∪g') = Fu‘f'+Fu‘g'.</u>

Hyp : (Sk)(k∈f') & (Sk)(k∈g') _____(1)

(1) : (Sk)(k∈f'∪g') _____(2)

(2), T.41 : k⌉⌊Fu‘f'∪g' ≡ (Al)(l∈f' v l∈g' ⊃ l⌉⌊k) _____(3)

(1), T.41 : k⌉⌊Fu‘f' ≡ (Al)(l∈f' ⊃ l⌉⌊k) _____(4)

(1), T.41 : k⌉⌊Fu‘g' ≡ (Al)(l∈g' ⊃ l⌉⌊k) _____(5)

(3), (4), (5) : k⌉⌊Fu‘f'∪g' ≡ k⌉⌊Fu‘f' & k⌉⌊Fu‘g' _____(6)

(6), T.32, T.22 : Fu‘f'∪g' = Fu‘f'+Fu‘g' _____(7)

(1), (7) : T.93.

94. $\underline{(Sk)(k \in f') \;\&\; \sim \mathrm{Fu}'f' = U \supset \overline{\mathrm{Fu}'\overline{f'}} \leq \mathrm{Fu}'\overline{f'}.}$

Hyp : $(Sk)(k \in f') \;\&\; \sim \mathrm{Fu}'f' = U$ ______ (1)

(1), T. 48, Defn. 2 : $(Sk) \sim (k \mathrm{o} \mathrm{Fu}'f')$ _____ (2)

(2), Defn. 4 : $(Sk)(k \rceil\!\lfloor \mathrm{Fu}'f')$ ______ (3)

Hyp : $(Ak)(k \in f')$ ______ (4)

(3), T. 41 : $(Sk)(Al)(l \in f' \supset l \rceil\!\lfloor k)$ _______ (5)

(4), (5) : $(Sk)(Al)(l \rceil\!\lfloor k)$ ______ (6)

(4), (6), T. 48 : $(Sk) \sim (k \in f')$ ______ (7)

(7) : $(Sk)(k \in \overline{f'})$ ______ (8)

Hyp : $k \rceil\!\lfloor \mathrm{Fu}'\overline{f'}$ ______ (9)

(8), (9), T. 41 : $(Al)(\sim l \in f' \supset l \rceil\!\lfloor k)$ _____ (10)

(10) : $l \mathrm{o} k \supset l \in f'$ _____ (11)

(11) : $k \mathrm{o} k \supset k \in f'$ ______ (12)

(12), T. 4, T. 39 : $k \leq \mathrm{Fu}'f'$ ______ (13)

(1), (13), T. 43 : $k \rceil\!\lfloor \overline{\mathrm{Fu}'\overline{f'}}$ _______ (14)

(9), (14), T. 21 : $\overline{\mathrm{Fu}'\overline{f'}} \leq \mathrm{Fu}'\overline{f'}$ ______ (15)

(1), (15) : T. 94.

95. $\underline{(Sl)(Ak)(k \in f' \vee k \in g' \supset l \leq k) \supset \mathrm{Nu}'(f' \cup g') = (\mathrm{Nu}'f')(\mathrm{Nu}'g').}$

Hyp : $(Sl)(Ak)(k \in f' \vee k \in g' \supset l \leq k)$ ______ (1)

(1) : $(Sl)(Ak)(k \in f' \supset l \leq k)$ ______ (2)

(1) : $(Sl)(Ak)(k \in g' \supset l \leq k)$ ______ (3)

(1), T. 61 : $k \leq \mathrm{Nu}'f' \cup g' \equiv (Al)(l \in f' \vee l \in g' \supset k \leq l)$ _____ (4)

(4) : $k \leq \mathrm{Nu}'f' \cup g' \equiv (Al)(l \in f' \supset k \leq l) \;\&\; (Al)(l \in g' \supset k \leq l)$ ____ (5)

(2), (3), (5), T. 61 : $k \leq \mathrm{Nu}'f' \cup g' \equiv k \leq \mathrm{Nu}'f' \;\&\; k \leq \mathrm{Nu}'g'$ ______ (6)

(1), (6), T.69 : $k \leq \text{Nu}^{\backprime} f' \vee g' \equiv k \leq (\text{Nu}^{\backprime} f')(\text{Nu}^{\backprime} g')$ ____(7)

(7), T.18 : $\text{Nu}^{\backprime} f' \vee g' = (\text{Nu}^{\backprime} f')(\text{Nu}^{\backprime} g')$ ____(8)

(1), (8) : T.95.

96. <u>$(Sl)(Ak)(k \in f' \supset l \leq k)$ & $(Sl)(Ak)(k \in g' \supset l \leq k) \supset \text{Nu}^{\backprime} f' + \text{Nu}^{\backprime} g'$ $\leq \text{Nu}^{\backprime}(f' \wedge g')$.</u>

Hyp : $(Sl)(Ak)(k \in f' \supset l \leq k)$ & $(Sl)(Ak)(k \in g' \supset l \leq k)$ ____(1)

(1) : $(Sl)(Ak)(k \in f'$ & $k \in g' \supset l \leq k)$ ____(2)

(1), T.63 : $\text{Nu}^{\backprime} f' \leq \text{Nu}^{\backprime}(f' \wedge g')$ ____(3)

(1), T.63 : $\text{Nu}^{\backprime} g' \leq \text{Nu}\ (f' \wedge g')$ ____(4)

(3), (4), T.37 : $\text{Nu}^{\backprime} f' + \text{Nu}^{\backprime} g' \leq \text{Nu}^{\backprime}(f' \wedge g')$ ____(5)

(1), (5) : T.96.

97. <u>$(S.h')(Ak)(k \in h' \equiv kok)$.</u>

A.3 : $(Sh')(Ak)(k \in h' \equiv kok)$ ____(1)

Hyp : $(Ak)(k \in h_1' \equiv kok)$ & $(Ak)(k \in h_2' \equiv kok)$ ____(2)

(2) : $(Ak)(k \in h_1' \equiv k \in h_2')$ ____(3)

(3), Defn.8 : $h_1' = h_2'$ ____(4)

(1), (2), (4) : T.97.

We now introduce the term $\vee$ for the unique h' such that $k \in h' \equiv kok$ and also the term $\wedge$ for $\bar{\vee}$.

98. <u>$\text{Fu}^{\backprime} \vee = U$.</u>

T.4 : $(Sk)(kok)$ ____(1)

(1), T.42 : $lo\text{Fu}^{\backprime} \vee \equiv (Sk)(kok$ & $kol)$ ____(2)

T.4 : $(Sk)(kok$ & $kol)$ ____(3)

(2), (3) : $(Al)(lo\text{Fu}^{\backprime} \vee)$ ____(4)

(4), T.51 : $Fu^{*}\mathcal{V}=U$.

We now define an atomic individual as follows : $At(k) =_{df} \sim(Sl)(l < k)$.

99. $At(k) \supset . l \leq k \supset l=k.$

Hyp : $At(k)$ ____(1)

(1), Defn.3, Defn.At : $(Al)(l \leq k \supset l=k)$ _____(2)

(2) : $l \leq k \supset l=k$ ____(3)

(1), (3) : T.99.

100. $At(k)$ & $At(l) \supset . k \rceil\lfloor l \vee k=l.$

Hyp : $At(k)$ & $At(l)$ _____(1)

Hyp : $k \circ l$ _____(2)

(2), T.2 : $m_1 \leq k$ & $m_1 \leq l$ _____(3) (m_1 is a constant.)

(1), (3), T.99 : $m_1=k$ & $m_1=l$ _____(4)

(4) : $k=l$ _____(5)

(2), (5) : $k \circ l \supset k=l$ ______(6)

(6), Defn.4 : $k \rceil\lfloor l \vee k=l$ ____(7)

(1), (7) : T.100.

If one wants to restrict the range of individual and set variables to a particular type of individual, then it can be carried out as follows :

$(Ak)\phi(k)$ becomes $(Ak)(k \in g' \supset \phi(k))$.

$(Sk)\phi(k)$ becomes $(Sk)(k \in g'$ & $\phi(k))$.

$(Af')\phi(f')$ becomes $(Af')(f' \subseteq g' \supset \phi(f'))$.

$(Sf')\phi(f')$ becomes $(Sf')(f' \subseteq g' \ \& \ \phi(f'))$.

The set g' must be non-empty.

The theory can easily be shown to be consistent as follows : Let there be only one individual, k, and two sets, Λ and $\{k\}$. Let kok and $k \in \{k\}$ be true and $k \in \Lambda$ be false. Axiom 1 becomes kok $\equiv$. kok $\supset$ kok & kok and is hence true. Axiom 2 becomes either $k \in \Lambda \supset kFu\Lambda$, which is true, or $k \in \{k\} \supset kFu\{k\}$, which is $k \in \{k\} \supset . k \rceil k \equiv . k \in \{k\} \supset k \rceil k$, which is true. Axiom 3 becomes $(Sf')(k \in f' \equiv \phi(k))$. $\phi(k)$ is true or false when particular substitutions are made for its free variables. If $\phi(k)$ is true then $k \in \{k\} \equiv \phi(k)$ is true and if $\phi(k)$ is false then $k \in \Lambda \equiv \phi(k)$ is true and hence $(Sf')(k \in f' \equiv \phi(k))$ is true. Axiom 4 becomes either $k=k \supset . k \in \Lambda \supset k \in \Lambda$ or $k=k \supset . k \in \{k\} \supset k \in \{k\}$. In both cases, they are true.

CHAPTER 4.

A 3-VALUED CLASS THEORY WITH INDIVIDUALS.

In this chapter, I wish to construct a class theory which is similar to NBG but has individuals as well as classes, and these individuals are used, as well as the null set, to generate classes. This theory differs from other class theories containing individuals in that it contains an axiomatic theory of individuals and uses a 3-valued significance logic in distinguishing between individuals and classes. On p.20 of [30], Suppes says, "However, our axioms do not actually postulate the existence of any individuals, and they are thus consistent with the view that there are only sets in the domain of discourse." So, he cannot ensure the existence of even one individual. Also he has to take care in distinguishing the null set from individuals. He can, however, construct classes from individuals by using his Axioms of Separation and Pairing.

The formal theory that follows will be similar to Mendelson's treatment of NBG in Chapter 4 of [17] except for certain modifications due to the presence of individuals or due to the use of the 3-valued significance logic. The formalisation is as follows :

Primitives.

1. u,v,w,x,y,z,.....(variables over special classes, i.e. the classes of this 3-valued theory, and individuals)

2. o(overlaps), ϵ (is a member of).

3. $\sim$, $\supset$, T_n (connectives of the 3-valued significance logic).

4. A, S (quantifiers of the 3-valued significance logic).

Formation Rules.

1. If x and y are variables then $x \circ y$ and $x \in y$ are atomic wffs.

2. If B and C are wffs then $\sim B$, $(B \supset C)$, $T_n B$ are wffs.

3. If B is a wff and x is a variable then $(Ax)B$ and $(Sx)B$ are wffs.

Definitions.

$Cl(x) =_{df} (Sy)S(y \in x)$. (x is a special class.)

$I(x) =_{df} \sim Cl(x)$. (x is an individual.)

Let us assume for the moment that there is at least one set (and hence at least one special class) and at least one individual.

$(Ak)\phi(k) =_{df} (Ax)(I(x) \supset \phi(x))$.

$(Sk)\phi(k) =_{df} (Sx)(T_n I(x) \;\&\; \phi(x))$.

Let k,l,m,n,....... be such individual variables.

$(Af)\phi(f) =_{df} (Ax)(Cl(x) \supset \phi(x))$.

$(Sf)\phi(f) =_{df} (Sx)(T_n Cl(x) \;\&\; \phi(x))$.

Let f,g,h,i,j,....... be such special class variables.

$M(f) =_{df} (Sg)(f \in g)$. (f is a set.)

$(Af')\phi(f') =_{df} (Af)(M(f) \supset \phi(f))$.

$(Sf')\phi(f') =df (Sf)(T_n M(f) \;\&\; \phi(f))$.

Let f',g',h',i',j',....... be such set variables.

$(Au')\phi(u') =_{df} (Au)(M(u) \vee I(u) \supset \phi(u))$.

$(Su')\phi(u') =_{df} (Su)(T_n(M(u) \vee I(u)) \;\&\; \phi(u))$.

Let u',v',w',x',y',z',....... be such variables over sets and individuals.

$x{=}y =_{df} (Ak)(kox \dot{\equiv} koy) \vee (Az)(z \in x \dot{\equiv} z \in y)$. (x is identical with y.) Notice, in this definition, that if x and y are classes then $x{=}y \cong (Az)(z \in x \equiv z \in y)$, if x and y are individuals then $x{=}y \cong (Ak)(kox \equiv koy)$, and if x is an individual and y a class, or vice versa, then x=y is non-significant. Notice also how the disjunction v is used to define identity over a range containing two different types of things. The definition could have been made without the use of v by taking each case in turn but it seems easier and more natural to use v.

I now give definitions restricted to individuals. In using these definitions, one cannot substitute one side of the definition for the other unless the variables x, y, etc. are restricted to individuals or sets(as the case may be).

$k \leq l =_{df} (Am)(mok \supset mol)$. (k is part of l.)

$k{=}l =_{df} (Am)(mok \equiv mol)$. (k is identical with l.)

[This is derived from the definition of x=y.]

$k < l =_{df} k \leq l \; \& \sim(k{=}l)$. (k is a proper part of l.)

$k \rceil\!\lfloor l =_{df} \sim(kol)$. (k is discrete from l.)

$kFuf' =_{df} (Am)(m \rceil\!\lfloor k \equiv (Al)(l \in f' \supset m \rceil\!\lfloor l))$. (k is the fusion of f'.)

$kNuf' =_{df} (Am)(m \leq k \equiv (Al)(l \in f' \supset m \leq l))$. (k is the nucleus of f'.)

I now give definitions restricted to special classes. Again, the definitions cannot be used unless the appropriate variable restrictions are made.

$f \subseteq g =_{df} (Az)(z \in f \supset z \in g)$. (f is a subclass of g.)

$f=g =_{df} (Az)(z \in f \equiv z \in g)$. (f is identical with g.)

[This is derived from the definition of $x=y$.]

$f \subset g =_{df} f \subseteq g \ \& \sim(f=g)$. (f is a proper subclass of g.)

If the variable restrictions are violated a non-significant wff may result. For example, $k \subseteq l$, $k \subset g$, $f \in g$, and $f \in k$ are all non-significant. Except in the case of xoy and $x \in y$, these non-significant wffs are avoided because it is simpler to use restricted variables and also the only purpose they could serve would be to form significance ranges but they do not introduce any new significance ranges which are not already obtained from xoy and $x \in y$.

<u>General Axioms.</u>

1. $S(x \in f)$.

2. $S(kol)$.

3. $Cl(x) \vee Cl(y) \supset \sim S(xoy)$.

<u>Individual Axioms.</u>

1. $kol \equiv (Sm)(An)(nom \supset nok \ \& \ nol)$.

2. $(Sk)(k \in f') \supset (Sl)(lFuf')$.

3. $(Sf')((Ak)(k \in f' \equiv \emptyset(k,l_1,\ldots,l_m)) \ \& \ (Ag')\sim(g' \in f'))$, where $\emptyset$ is constructed using only o, ~, &, A and variables quantified over individuals.

4. $k=l \supset . k \in f \equiv l \in f$.

5. $(Sx)I(x)$.

<u>Special Class Axioms.</u>

T. $f=g \supset (Ah)(f \in h \equiv g \in h)$.

P. $(Ax')(Ay')(Sf')(Au')(u'\in f' \equiv T(u'=x' \vee u'=y'))$.

N. $(Sf')(Ax')\sim x'\in f'$.

U. $(Af')(Sg')(Ax')(x'\in g' \equiv (Sh')(x'\in h' \ \& \ h'\in f'))$.

W. $(Af')(Sg')(Ax')(x'\in g' \equiv T(Ay')(y'\in x' \supset y'\in f'))$.

S. $(Af')(Ag)(Sg')(Ax')(x'\in g' \equiv x'\in f' \ \& \ x'\in g)$.

Before introducing more axioms, I need to prove some theorems and introduce more definitions.

T.1. $\underline{\sim S(x\in k)}$.

Defn.I : $T(y) \supset (Ax)\sim S(x\in y)$ _____(1)

(1), Defn.k : $\sim S(x\in k)$,

T.2. $\underline{k=1 \supset .\emptyset(k) \simeq \emptyset(1)}$, for any wff $\emptyset$.

Gen.Ax.2, Defn.= : $k=1 \supset .mok \simeq mol$ ______(1)

Gen.Ax.2, (1) : $k=1 \supset .kom \simeq lom$ ______(2)

Gen.Ax.1, Ind.Ax.4 : $k=1 \supset .k\in f \simeq 1\in f$ ______(3)

T.1 : $k=1 \supset .x\in k \simeq x\in 1$ _____(4)

T.1 : $k=1 \supset .k\in m \simeq 1\in m$ _____(5)

Gen.Ax.3 : $k=1 \supset .fok \simeq fol$ _____(6)

Gen.Ax.3 : $k=1 \supset .kof \simeq lof$ ______(7)

By using induction on the number of connectives and quantifiers in $\emptyset$, T.2 can now be shown.

T.3. $\underline{f=g \equiv .f\subseteq g \ \& \ g\subseteq f}$.

Defns.=, $\subseteq$: T.3.

T.4. $\underline{f=f}$.

Defn.= : T.4.

T.5. $f=g \supset g=f.$

Defn.= : T.5.

T.6. $f=g \supset. g=h \supset f=h.$

Defn.= : $f=g \equiv (Az)(z\in f \equiv z\in g)$ ______(1)

Defn.= : $g=h \equiv (Az)(z\in g \equiv z\in h)$ ______(2)

(1), (2) : $f=g \supset. g=h \supset (Az)(z\in f \equiv z\in h)$ _____(3)

(3), Defn.= : T.6.

T.7. $f=g \supset. \phi(f) \simeq \phi(g)$, for any wff ϕ.

Gen.Ax.1, Defn.= : $f=g \supset. z\in f \simeq z\in g$ _____(1)

Gen.Ax.1, Ax.T : $f=g \supset. f\in h \simeq g\in h$ _____(2)

T.1 : $f=g \supset. f\in k \simeq g\in k$ _____(3)

Gen.Ax.3 : $f=g \supset. f o x \simeq g o x$ _____(4)

Gen.Ax.3 : $f=g \supset. x o f \simeq x o g$ _____(5)

By using induction on the number of connectives and quantifiers in ϕ, T.7 can now be shown.

Define a proper class as a special class which is not a set.

$Pr(f) =_{df} \sim M(f).$

T.8. $Pr(f) \supset F(f\in g).$

Defns.Pr, M : T.8.

T.9. $(Ax')(Ay')(S!f')(Au')(u'\in f' \equiv T(u'=x' \vee u'=y')).$

Ax.P : $(Ax')(Ay')(Sf')(Au')(u'\in f' \equiv T(u'=x' \vee u'=y'))$ ____(1)

Hyp : $(Au')(u'\in f_1' \equiv T(u'=x' \vee u'=y'))$ & $(Au')(u'\in f_2' \equiv T(u'=x' \vee u'=y'))$ _____(2)

(2) : $(Au')(u'\in f_1' \equiv u'\in f_2')$ _____(3)

(3), T.8 : $(Au)(u \in f_1' \equiv u \in f_2')$ ______(4)

(4), Defn.= : $f_1'=f_2'$ ______(5)

(1), (2), (5) : T.9.

We now introduce the definition $\{x',y'\}$ (the unordered pair of x' and y') for the unique set f' such that $(Au')(u' \in f' \equiv T(u'=x' \vee u'=y'))$. Also define $\{x'\}$ as $\{x',x'\}$.

T.10. $(Au')(u' \in \{x'\} \equiv T(u'=x'))$.

Defn. $\{x'\}$: $(Au')(u' \in \{x'\} \equiv T(u'=x' \vee u'=x'))$ ______(1)

(1) : T.10.

T.11. $\{x',y'\}=\{y',x'\}$.

Defn. $\{x',y'\}$: T.11.

T.12. $\{x'\}=\{y'\} \supset. x'=y'$.

Hyp : $\{x'\}=\{y'\}$ ______(1)

(1), Defn.= : $u' \in \{x'\} \equiv u' \in \{y'\}$ ______(2)

(2), T.10 : $(Au')(T(u'=x') \equiv T(u'=y'))$ ______(3)

Hyp : $I(x')$ & $I(y')$ ______(4)

(3), Defn.u' : $(Ak)(T(k=x') \equiv T(k=y'))$ ______(5)

(5) : $x'=y'$ ______(6)

Hyp : $Cl(x')$ & $Cl(y')$ ______(7)

(3), Defn.u' : $(Af')(T(f'=x') \equiv T(f'=y'))$ ______(8)

(8) : $x'=y'$ ______(9)

Hyp : $(I(x') \& Cl(y')) \vee (Cl(x') \& I(y'))$ ______(10)

(10), Defn.= : $\sim S(x'=y')$ ______(11)

(10), (3) : $F(\{x'\}=\{y'\})$ ______(12)

(1), (4), (6), (7), (9), (10), (11), (12) : T.12.

T.13. $\underline{(S!f')(Ax')\sim x'\in f'.}$

Ax.N : $(Sf')(Ax')\sim x'\in f'$ _____(1)

Hyp : $(Ax')\sim x'\in f_1'$ & $(Ax')\sim x'\in f_2'$ ____(2)

(2) : $(Ax')(x'\in f_1' \equiv x'\in f_2')$ ______(3)

(3), T.8 : $(Ax)(x\in f_1' \equiv x\in f_2')$ _______(4)

(4), Defn.= : $f_1'=f_2'$ ______(5)

(1), (2), (5) : T.13.

We now introduce the definition 0 (the null set) for the unique set f' such that $(Ax')\sim x'\in f'$. We now have at least one set as required for the definition of set and special class variables. Individual Axiom 5 ensures the existence of at least one individual for the definition of individual variables.

We define an ordered pair, $\langle x',y'\rangle$, of x' and y' as $\{\{x'\},\{x',y'\}\}$.

T.14. $\underline{\langle x',y'\rangle=\langle u',v'\rangle \supset x'=u' \;\&\; y'=v'.}$

Hyp : $\langle x',y'\rangle=\langle u',v'\rangle$ ______(1)

(1), Defn. $\langle x',y'\rangle$: $\{\{x'\},\{x',y'\}\}=\{\{u'\},\{u',v'\}\}$ ______(2)

(2), Defns. $\{x',y'\}$,= : $\{x'\}\in\{\{u'\},\{u',v'\}\}$ _______(3)

(3), Defn. $\{x',y'\}$: $\{x'\}=\{u'\}$ v $\{x'\}=\{u',v'\}$ ______(4)

(4), T.12, Defn. $\{x'\}$: $x'=u'$ ______(5)

(2), Defns. $\{x',y'\}$,= : $\{u',v'\}\in\{\{x'\},\{x',y'\}\}$ ______(6)

(6), Defn. $\{x',y'\}$: $\{u',v'\}=\{x'\}$ v $\{u',v'\}=\{x',y'\}$ ______(7)

Similarly, $\{x',y'\}=\{u'\}$ v $\{x',y'\}=\{u',v'\}$ ______(8)

Defn. $\{x'\}$: $\{u',v'\}=\{x'\}$ & $\{x',y'\}=\{u'\} \supset y'=v'$ ______(9)

Hyp : $\{u',v'\}=\{x',y'\}$ _____(10)

(5), (10) : $\{u',v'\}=\{u',y'\}$ _____(11)

(10), Defn. $\{x',y'\}$: $T(y'=u')\ v\ T(y'=v')$ _____(12)

(12) : $u'=v' \supset T(y'=v')$ _____(13)

(11), Defns. $\{x',y'\}$, = : $y'=u' \supset T(u'=v')$ _____(14)

(12), (14) : $\sim T(u'=v') \supset T(y'=v')$ _____(15)

(10), (13), (15) : $\{u',v'\}=\{x',y'\} \supset T(y'=v')$ _____(16)

(7), (8), (9), (16) : $T(y'=v')$ _____(17)

(5), (17) : $x'=u'$ & $y'=v'$ _____(18)

(1), (18) : T.14.

The definition of ordered pairs can be extended as follows :

$\langle x'\rangle =_{df} x$, $\langle x_1',\ldots,x_{n+1}'\rangle =_{df} \langle\langle x_1',\ldots,x_n'\rangle, x_{n+1}'\rangle$.

T.15. $\underline{\langle x_1',\ldots,x_n'\rangle=\langle y_1',\ldots,y_n'\rangle \supset . x_1'=y_1' \ \& \ \ldots . \ \& \ x_n'=y_n'}$.

Hyp : $\langle x_1',\ldots,x_n'\rangle=\langle y_1',\ldots,y_n'\rangle$ _____(1)

T.14, Defn. $\langle x_1',\ldots,x_n'\rangle$: $\langle x_1',\ldots,x_{n-1}'\rangle=\langle y_1',\ldots,y_{n-1}'\rangle$ & $x_n'=y_n'$ _____(2)

By T.14, Defn. $\langle x_1',\ldots,x_n'\rangle$, using induction on n, $x_1'=y_1'$ & .... $x_n'=y_n'$ _____(3)

(1), (3) : T.15.

(see next page)

T.16. $\underline{(Af')(S!g')(Ax')(x'\epsilon g' \equiv (Sh')(x'\epsilon h' \ \& \ h'\epsilon f')).}$

Ax.U : $(Sg')(Ax')(x'\epsilon g' \equiv (Sh')(x'\epsilon h' \ \& \ h'\epsilon f'))$ ____(1)

Hyp : $(Ax')(x'\epsilon g_1' \equiv (Sh')(x'\epsilon h' \ \& \ h'\epsilon f')) \ \& \ (Ax')(x'\epsilon g_2' \equiv (Sh')(x'\epsilon h' \ \& \ h'\epsilon f'))$ ____(2)

(2) : $(Ax')(x'\epsilon g_1' \equiv x'\epsilon g_2')$ ____(3)

(3), T.8, Defn. = : $g_1'=g_2'$ ____(4)

(1), (2), (4) : T.16.

We now introduce the definition U(f') (the sum set of f') for the unique g' such that $(Ax')(x'\epsilon g' \equiv (Sh')(x'\epsilon h' \ \& \ h'\epsilon f'))$. Also define $f'\cup g'$ as $U(\{f',g'\})$.

T.17. $\underline{(Ax')(x'\epsilon f'\cup g' \equiv x'\epsilon f' \ v \ x'\epsilon g').}$

Defns. $\cup$ and U : $x'\epsilon f'\cup g' \equiv (Sh')(x'\epsilon h' \ \& \ h'\epsilon\{f',g'\})$ ____(1)

(1), Defn. { } : $x'\epsilon f'\cup g' \equiv (Sh')(x'\epsilon h' \ \&. T(h'=f') \ v \ T(h'=g'))$ ____(2)

(2) : $x'\epsilon f'\cup g' \equiv x'\epsilon f' \ v \ x'\epsilon g'$.

We can now define $\{x_1',\dots,x_n'\}$ inductively as $\{x_1',\dots,x_{n-1}'\}\cup\{x_n'\}$.

T.18. $\underline{(Au')(u'\epsilon\{x_1',x_2',\dots,x_n'\} \equiv T(u'=x_1') \ v - v \ T(u'=x_n').}$

T.10 : $u'\epsilon\{x_1'\} \equiv T(u'=x_1')$ ____(1)

Defn. { } : $u'\epsilon\{x_1',x_2'\} \equiv T(u'=x_1') \ v \ T(u'=x_2')$ ____(2)

T.17, Defn. { } : $u'\epsilon\{x_1',\dots,x_{n+1}\} \equiv u'\epsilon\{x_1',\dots,x_n'\} \ v \ u'\epsilon\{x_{n+1}'\}$ ____(3)

(3), T.10 : $u'\epsilon\{x_1',\dots,x_{n+1}'\} \equiv u'\epsilon\{x_1',\dots,x_n'\} \ v \ T(u'=x_{n+1}')$ ____(4)

By induction, using (1), (2), (4), T,18 follows.

T.19. $\underline{(Af')(S!g')(Ax')(x'\epsilon g' \equiv T(Ay')(y'\epsilon x' \supset y'\epsilon f')).}$

Ax.W : $(Sg')(Ax')(x'\epsilon g' \equiv T(Ay')(y'\epsilon x' \supset y'\epsilon f'))$ ____(1)

Hyp : $(Ax')(x'\epsilon g_1' \equiv T(Ay')(y'\epsilon x' \supset y'\epsilon f')) \mathbin{\&} (Ax')(x'\epsilon g_2' \equiv T(Ay')(y'\epsilon x' \supset y'\epsilon f'))$ ____(2)

(2) : $(Ax')(x'\epsilon g_1' \equiv x'\epsilon g_2')$ ____(3)

(3), T.8, Defn. = : $g_1'=g_2'$ ____(4)

(1), (2), (4) : T.19.

We now define the power set of the set f', $\mathcal{P}(f')$, as the unique g' such that $(Ax')(x'\epsilon g' \equiv T(Ay')(y'\epsilon x' \supset y'\epsilon f'))$.

T.20. $(Af')(Ag)(S!g')(Ax')(x'\epsilon g' \equiv . x'\epsilon f' \mathbin{\&} x'\epsilon g)$.

Ax.S : $(Sg')(Ax')(x'\epsilon g' \equiv . x'\epsilon f' \mathbin{\&} x'\epsilon g)$ ____(1)

Hyp : $(Ax')(x'\epsilon g_1' \equiv . x'\epsilon f' \mathbin{\&} x'\epsilon g) \mathbin{\&} (Ax')(x'\epsilon g_2' \equiv . x'\epsilon f' \mathbin{\&} x'\epsilon g)$ ____(2)

(2) : $(Ax')(x'\epsilon g_1' = x'\epsilon g_2')$ ____(3)

(3), T.8, Defn. = : $g_1'=g_2'$ ____(4)

(1), (2), (4) : T.20.

We now define the intersection set of the set f' and the class g, $f'\cap g$, as the unique g' such that $(Ax')(x'\epsilon g' \equiv . x'\epsilon f' \mathbin{\&} x'\epsilon g)$.

T.21. $f \subseteq f' \supset M(f)$.

Defn.∩ : $x'\epsilon f\cap f' \equiv x'\epsilon f \mathbin{\&} x'\epsilon f'$ ____(1)

Hyp : $f \subset f'$ ____(2)

(2), Defn. ⊆ : $x'\epsilon f \subset x'\epsilon f'$ ____(3)

(1), (3) : $x'\epsilon f\cap f' \equiv x'\epsilon f$ ____(4)

(4), T.8 : $f\cap f'=f$ ____(5)

Defn. ∩ : $M(f\cap f')$ ____(6)

(5), (6) : $M(f)$ ____(7)

(2), (7) : T.21.

In preparation for the next set of axioms we need the definition of a univocal class (relation) : $Un(f) =_{df} (Ax')(Ay')(Az')(\langle x',y'\rangle \epsilon f \;\&\; \langle x',z'\rangle \epsilon f \supset T(y'=z'))$.

Further Special Class Axioms.

B. $(Ax_1',\ldots,x_\ell',y_1,\ldots,y_m)S\phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m) \supset (Sf)$ $(Ax_1',\ldots,x_\ell')(\langle x_1',\ldots,x_\ell'\rangle \epsilon f \equiv \phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m))$, where ϕ is constructed using 0, ϵ, $\sim$, $\supset$, T_n, A, S (quantifier) such that only variables over sets and individuals are quantified, and $x_1',\ldots,x_\ell',y_1,\ldots,y_m$ are all the free variables of ϕ and f is not amongst them.

R. $(Af')(Un(f) \supset (Sg')(Ax')(x'\epsilon g' \equiv (Sy')(\langle y',x'\rangle \epsilon f \;\&\; y'\epsilon f')))$.

I. $(Sf')(0\epsilon f' \;\&\; (Ag')(g'\epsilon f' \supset g'\cup\{g'\}\epsilon f'))$.

T.22. $(S!h)(Au')(u'\epsilon h \equiv (Sv')(Sw')(T(u'=\langle v',w'\rangle) \;\&\; v'\epsilon f \;\&\; w'\epsilon g))$.

Ax.B : $(Sh)(Au')(u'\epsilon h \equiv (Sv')(Sw')(T(u'=\langle v',w'\rangle) \;\&\; v\epsilon f \;\&\; w'\epsilon g))$ ____ (1)

Hyp : $(Au')(u'\epsilon h_1 \equiv (Sv')$-----$) \;\&\; (Au')(u'\epsilon h_2 \equiv (Sv')$-----$)$ ____ (2)

(2) : $(Au')(u'\epsilon h_1 \equiv u'\epsilon h_2)$ ____ (3)

(3), T.8, Defn. = : $h_1=h_2$ ____ (4)

(1), (2), (4) : T.22.

We now introduce the definition fxg (the Cartesian product of classes f and g) as the unique h such that $(Au')(u'\epsilon h \equiv (Sv')(Sw')(T(u'=\langle v',w'\rangle) \;\&\; v'\epsilon f \;\&\; w'\epsilon g)$.

Let f^2 be defined as fxf and f^n be defined as f^{n-1}xf.

T.23. $(S!h)(Au')(u'\epsilon h \equiv u'\epsilon f \;\&\; u'\epsilon g)$.

Proof is similar is that of T.22. Define $f\cap g$ (the intersection of classes f and g) as the unique h such that T.23 holds.

T.24. $\underline{(S!h)(A\mathfrak{u}')(\mathfrak{u}'\epsilon h \equiv \mathfrak{u}'\epsilon f \vee \mathfrak{u}'\epsilon g).}$

Proof is similar to that of T.22. Define $f \cup g$ (the union of classes f and g) as the unique h such that T.24 holds.

T.25. $\underline{(S!g)(A\mathfrak{u}')(\mathfrak{u}'\epsilon g \equiv \sim\mathfrak{u}'\epsilon f).}$

Proof is similar to that of T.22. Define $\overline{f}$ (the complement of the class f) as the unique g such that T.25 holds. Also define f-g as $f \cap \overline{g}$.

T.26. $\underline{(S!f)(A\mathfrak{u}')(\mathfrak{u}'\epsilon f \equiv \mathfrak{u}'=\mathfrak{u}').}$

Proof is similar to that of T.22. Define V (the universal class) as the unique f such that T.26 holds.

T.27. $\underline{(S!g)(A\mathfrak{u}')(\mathfrak{u}'\epsilon g \equiv (Sv')(<\mathfrak{u}',v'>\epsilon f)).}$

Proof is similar to that of T.22. Define $\mathcal{D}(f)$ (the domain of f) as the unique g such that T.27 holds.

T.28. $\underline{(S!g)(A\mathfrak{u}')(\mathfrak{u}'\epsilon g \equiv (Sv')(<v',\mathfrak{u}'>\epsilon f)).}$

Proof is similar to that of T.22. Define $\mathcal{R}(f)$ (the range of f) as the unique g such that T.28 holds.

T.29. $\underline{\overline{f \cup g} = \overline{f} \cap \overline{g}.}$

Defns. $\cup, \overline{\ }$: $\mathfrak{u}'\epsilon\overline{f \cup g} \equiv \sim(\mathfrak{u}'\epsilon f \vee \mathfrak{u}'\epsilon g)$ ____(1)

(1), Defns. $\cap, \overline{\ }$: $\mathfrak{u}'\epsilon\overline{f \cup g} \equiv \mathfrak{u}'\epsilon\overline{f} \cap \overline{g}$ ____(2)

(2), T.8, Defn. = : T.29.

T.30. $\underline{\overline{\overline{f}}=f.}$

Defn. $\overline{\ }$: $\mathfrak{u}'\epsilon\overline{\overline{f}} \equiv \sim\sim(\mathfrak{u}'\epsilon f)$ ____(1)

(1), T.8, Defn. = : $\overline{\overline{f}}=f$.

T.31. $\underline{V=\overline{0}.}$

Defns. $0, \bar{\ }$: $\mathfrak{u}'\in\overline{0}$ ____(1)

(1) : $\mathfrak{u}'\in\overline{0} \equiv \mathfrak{u}'=\mathfrak{u}'$ ____(2)

(2), T.8. Defns. $V, =$: $V=\overline{0}$.

T.32. $\underline{(A\mathfrak{u}')(\mathfrak{u}'\in V)}$.

Defn. V : $\mathfrak{u}'\in V \equiv \mathfrak{u}'=\mathfrak{u}'$ ____(1)

(1) : $(A\mathfrak{u}')(\mathfrak{u}'\in V)$.

T.33. $\underline{(A\mathfrak{u}')(\mathfrak{u}'\in f\text{-}g \equiv \mathfrak{u}'\in f \;\&\; {\sim}\mathfrak{u}'\in g)}$.

Defn. - : $\mathfrak{u}'\in f\text{-}g \equiv \mathfrak{u}'\in f\cap\overline{g}$ ____(1)

(1), Defns. $\cap, \bar{\ }$: $\mathfrak{u}'\in f\text{-}g \equiv \mathfrak{u}'\in f \;\&\; {\sim}\mathfrak{u}'\in g$.

The following theorems follow trivially from the definitions:

T.34. $f\cap g=g\cap f$.

T.35. $f\cap(g\cap h)=(f\cap g)\cap h$.

T.36. $f\cap f=f$.

T.37. $f\cap 0=0$.

T.38. $f\cap V=f$.

T.39. $f\cup g=g\cup f$.

T.40. $f\cup(g\cup h)=(f\cup g)\cup h$.

T.41. $f\cup f=f$.

T.42. $f\cup 0=f$.

T.43. $f\cup V=V$.

T.44. $f\cap(g\cup h)=(f\cap g)\cup(f\cap h)$.

T.45. $f\cup(g\cap h)=(f\cup g)\cap(f\cup h)$,

T.46. $\overline{f\cap g}=\overline{f}\cup\overline{g}$.

T.47. $f\text{-}f=0$.

T.48. $V-f=\overline{f}$.

T.49. $\overline{V}=0$.

We now define a relation as follows:

$\mathrm{Rel}(f) =_{df} f \subseteq V^2$.

T.50. $\underline{(S!g)(Au')(u'\epsilon g \equiv T(Av')(v'\epsilon u' \supset v'\epsilon f))}$.

Proof is similar to that of T.22. Define $\wp(f)$ (the power class of f) as the unique g such that T.50 holds.

T.51. $\underline{(S!g)(Au')(u'\epsilon g \equiv (Sh')(u'\epsilon h' \;\&\; h'\epsilon f))}$.

Proof is similar to that of T.22. Define U(f) (the sum class of f) as the unique g such that T.51 holds.

T.52. $\underline{(S!f)(Au')(u'\epsilon f \equiv T(Sv')(u'=<v',v'>))}$.

Proof is similar to that of T.22. Define I (the identity relation) as the unique f such that T.52 holds.

T.53. $\underline{(Ax_1',\ldots,x_\ell',y_1,\ldots,y_m)S\phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m) \supset (S!f)}$
$\underline{(f \subseteq V^\ell \;\&\; (Ax_1',\ldots,x_\ell')(<x_1',\ldots,x_\ell'>\epsilon f \equiv \phi(x_1',\ldots,x_\ell',}$
$\underline{y_1,\ldots,y_m)))}$, where ϕ is constructed as in Axiom B.

Hyp : $(Ax_1',\ldots,x_\ell',y_1,\ldots,y_m)S\phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m)$ ____(1)

(1), Ax.B : $(Ax_1',\ldots,x_\ell')(<x_1',\ldots,x_\ell'>\epsilon f_1 \equiv \phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m))$
____(2) (f_1 is a constant)

(2), Defn. V^ℓ : $f_1 \cap V^\ell \subseteq V^\ell \;\&\; (Ax_1',\ldots x_\ell')(<x_1',\ldots,x_\ell'>\epsilon f_1 \cap V^\ell \equiv$
$\phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m))$ ____(3)

Hyp : $g_1 \subseteq V^\ell \;\&\; (Ax_1',\ldots,x_\ell')(<x_1',\ldots,x_\ell'>\epsilon g_1 \equiv \phi(x_1',\ldots,x_\ell',$
$y_1,\ldots,y_m)) \;\&\; g_2 \subseteq V^\ell \;\&\; (Ax_1',\ldots,x_\ell')(<x_1',\ldots,x_\ell'>\epsilon g_2 \equiv$
$\phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m))$ ____(4)

(4) : $<x_1', \ldots, x_\ell'> \in g_1 \equiv <x_1', \ldots, x_\ell'> \in g_2$ ____(5)

(5), Defn. V^ℓ : $(Au')(u' \in V^\ell \supset . u' \in g_1 \equiv u' \in g_2)$ ____(6)

(4) : $(Au')(\sim u' \in V^\ell \supset . u' \in g_1 \equiv u' \in g_2)$ ____(7)

(6), (7), T.8 : $g_1 = g_2$ ____(8)

(1), (3), (4), (8) : T.53.

Define $\{<x_1', \ldots, x_\ell'> / \phi(x_1', \ldots, x_\ell', y_1, \ldots, y_m)\}$ (the class of ordered ℓ-tuples such that ϕ holds) as the unique f such that T.53 holds.

Define the inverse relation of f, $\breve{f}$, as $\{<x_1', x_2'> / <x_2', x_1'> \in f\}$.

T.54. $\underline{\mathcal{R}(f) = \mathcal{D}(\breve{f})}$.

Defn. $\mathcal{R}$: $u' \in \mathcal{R}(f) \equiv (Sv')(<v', u'> \in f)$ ____(1)

Defn. $\mathcal{D}$: $u' \in \mathcal{D}(f) \equiv (Sv')(<u', v'> \in f)$ ____(2)

Defn. $\breve{}$: $<u', v'> \in f \equiv <v', u'> \in \breve{f}$ ____(3)

(1), (2), (3) : $u' \in \mathcal{R}(f) \equiv u \in \mathcal{D}(\breve{f})$ ____(4)

(4), T.8, Defn. = : T.54.

We now define the following:

$Fnc(f) =_{df} Rel(f) \,\&\, Un(f)$ (f is a function).

$g1f =_{df} f \cap (g \times V)$ (the restriction of f to the domain g).

$Un_1(f) =_{df} Un(f) \,\&\, Un(\breve{f})$ (f is one-to-one).

If there is a unique z' such that $<y', z'> \in f$ then $f`y' =_{df} z'$.

$f``g = \mathcal{R}(g1f)$.

T.55. $\underline{(Af')(Un(f) \supset (S!g')(Ax')(x' \in g' \equiv (Sy')(<y', x'> \in f \,\&\, y' \in f')))}$.

Proof is similar to that of T.22. Define the set, $\mathcal{R}(f'1f)$, as the unique g' such that T.55 holds.

T.56. $\vdash$ Axiom R $\Rightarrow$ $\vdash$ Axiom S.

Hyp : Axiom R ____(1)

Defn. Un : $Un(\{<x_1', x_2'> \mid T(x_1'=x_2') \ \& \ x_1' \epsilon h\})$ ____(2)

(1), (2) : $(Sg')(Ax')(x' \epsilon g' \equiv (Sy')(T(y'=x') \ \& \ y' \epsilon h \ \& \ y' \epsilon f'))$ ____(3)

(3) : $(Sg')(Ax')(x' \epsilon g' \equiv x' \epsilon h \ \& \ x' \epsilon f')$ ____(4)

T.57. $M(\mathfrak{D}(f'))$.

Hyp : $\mathfrak{u}' \epsilon \mathfrak{D}(f')$ ____(1)

(1), Defn. $\mathfrak{D}$: $(Sv')(<\mathfrak{u}', v'> \epsilon f')$ ____(2)

Defns. < >, U : $\{\mathfrak{u}'\} \epsilon U(f')$ ____(3)

(3), Defns. { }, U : $\mathfrak{u}' \epsilon U(U(f'))$ ____(4)

(1), (4) : $\mathfrak{D}(f') \subseteq U(U(f'))$ ____(5)

(5), T.21 : $M(\mathfrak{D}(f'))$.

T.58. $M(\mathfrak{R}(f'))$.

Proof is similar to that of T.57.

T.59. $M(f'xg')$.

Hyp : $\mathfrak{u}' \epsilon f'xg'$ ____(1)

(1), Defn. x : $T(\mathfrak{u}'=<v_1', w_1'>) \ \& \ v_1' \epsilon f' \ \& \ w_1' \epsilon g')$ ____(2) (v_1 and w_1 are constants)

(2) : $\{v_1'\} \subseteq f' \cup g' \ \& \ \{v_1', w_1'\} \subseteq f' \cup g'$ ____(3)

(3), Defn. $\mathfrak{P}$: $\{v_1'\} \epsilon \mathfrak{P}(f' \cup g') \ \& \ \{v_1', w_1'\} \epsilon \mathfrak{P}(f' \cup g')$ ____(4)

(4), Defns. $\mathfrak{P}$, < > : $<v_1', w_1'> \epsilon \mathfrak{P}(\mathfrak{P}(f' \cup g'))$ ____(5)

(2), (5) : $\mathfrak{u}' \epsilon \mathfrak{P}(\mathfrak{P}(f' \cup g'))$ ____(6)

(1), (6) : $f'xg' \subseteq \mathfrak{P}(\mathfrak{P}(f' \cup g'))$ ____(7)

(7), T.21 : $M(f'xg')$.

T.60. $\underline{M(\mathfrak{D}(f))\ \&\ M(\mathfrak{R}(f))\ \&\ \mathrm{Rel}|\mathfrak{f}\ (f) \supset M(f).}$

Hyp : $M(\mathfrak{D}(f))\ \&\ M(\mathfrak{R}(f))\ \&\ \mathrm{Rel}|\mathfrak{f}\ (f)$ ____(1)

Hyp : $\mathfrak{u}' \in f$ ____(2)

(1), (2) : $\mathfrak{u}' = \langle v_1', w_1' \rangle$ ____(3) (v_1', w_1' are constants).

(2), (3), Defns. $\mathfrak{D}$, $\mathfrak{R}$: $v_1' \in \mathfrak{D}(f)\ \&\ w_1' \in \mathfrak{R}(f)$ ____(4)

(3), (4) : $(Sv')(Sw')(\mathfrak{u}' = \langle v', w' \rangle\ \&\ v' \in \mathfrak{D}(f)\ \&\ w' \in \mathfrak{R}(f))$ ____(5)

(5), Defn. x : $\mathfrak{u}' \in \mathfrak{D}(f) \times \mathfrak{R}(f)$ ____(6)

(2), (6) : $f \subseteq \mathfrak{D}(f) \times \mathfrak{R}(f)$ ____(7)

(1), (7), T.59, T.21 : $M(f)$ ____(8)

(1), (8) : T.60.

T.61. $\underline{\mathrm{Fnc}(f) \supset M(f'1f).}$

Hyp : $\mathrm{Fnc}(f)$ ____(1)

Defns. $\mathrm{Rel}|\mathfrak{f}$, 1 : $\mathrm{Rel}|\mathfrak{f}\ (f'1f)$ ____(2)

Defns. Un, 1 : $\mathrm{Un}(f'1f)$ ____(3)

(2), (3), Defn. Fnc : $\mathrm{Fnc}(f'1f)$ ____(4)

Defns. $\mathfrak{D}$, 1 : $\mathfrak{D}(f'1f) \subseteq f'$ ____(5)

(5), T.21 : $M(\mathfrak{D}(f'1f))$ ____(6)

(3), (6), Ax.R : $M(\mathfrak{R}(f'1(f'1f)))$ ____(7)

(7), Defn. 1 : $M(\mathfrak{R}(f'1f))$ ____(8)

(2), (6), (8), T.60 : $M(f'1f)$ ____(9)

(1), (9) : T.61.

T.62. $\underline{\mathrm{Pr}(V).}$

Ax.B : $(A\mathfrak{u}')(\mathfrak{u}' \in f_1 \equiv T{\sim}\mathfrak{u}' \in \mathfrak{u}')$ ____(1) (f_1 is a constant).

(1) : $(A\mathfrak{u})(M(\mathfrak{u}) \supset . \mathfrak{u} \epsilon f_1 \equiv T \sim \mathfrak{u} \epsilon \mathfrak{u})$ ____(2)

(2) : $M(f_1) \supset . f_1 \epsilon f_1 \equiv F(f_1 \epsilon f_1)$ ____(3)

3, Defn. Pr : $Pr(f_1)$ ____(4)

T.32 : $f_1 \subseteq V$ ____(5)

(4), (5), T.21 : $Pr(V)$.

T.63. <u>M(I).</u>

Indiv. Ax.3 : $(Sf')((Ak)(k \epsilon f' \equiv kok) \,\&\, (Ag')\sim(g' \epsilon f'))$ ____(1)

Let I be the set of all individuals, defined by (1), f' being unique by T.8 and the Defn. = . Hence, M(I).

T.64. <u>$(A\mathfrak{u}', y_1, \ldots, y_m)S\phi(\mathfrak{u}', y_1, \ldots, y_m) \supset (S!f')(Ak)(k \epsilon f' \equiv \phi(k, y_1, \ldots, y_m)) \,\&\, (Ag')\sim(g' \epsilon f'))$, where ϕ is any wff, not containing f' free and containing quantification over set and individual variables only, and where $k, y_1, \ldots, y_m$ are all the free variables in ϕ.</u>

Hyp : $(A\mathfrak{u}', y_1, \ldots, y_m)S\phi(\mathfrak{u}', y_1, \ldots, y_m)$ ____(1)

(1), Ax.B : $(S!f)(A\mathfrak{u}')(\mathfrak{u}' \epsilon f \equiv \phi(\mathfrak{u}', y_1, \ldots, y_m) \,\&\, \sim(Sy')S(y' \epsilon \mathfrak{u}'))$ ____(2)

(2), T.8 : $(S!f)(Ak)(k \epsilon f \equiv \phi(k, y_1, \ldots, y_m)) \,\&\, (Ag')\sim(g' \epsilon f))$ ____(3)

(3), Defn. V : $f_1 \subseteq V$ ____(4) (f_1 is the unique f of (3).)

(4), T.63, T.21 : $M(f_1)$ ____(5)

(1), (3), (5) : T.64.

T.64 corresponds to Axiom 3 of the last chapter, where any wff could be substituted for ϕ.

From this point onwards, I will only state the theorems and only give proofs where they differ from those of NBG in Mendelson's book.

$f\mathrm{Irr}g =_{df} \mathrm{Rel}(f) \;\&\; (Ay')(y' \epsilon g \supset \sim\langle y',y'\rangle \epsilon f)$. (f is an irreflexive relation on g).

$f\mathrm{Tr}g =_{df} \mathrm{Rel}(f) \;\&\; (Au')(Av')(Aw')(u' \epsilon g \;\&\; v' \epsilon g \;\&\; w' \epsilon g \;\&\; \langle u',v'\rangle \epsilon f \;\&\; \langle v',w'\rangle \epsilon f \supset \langle u',w'\rangle \epsilon f)$. (f is a transitive relation on g).

$f\mathrm{Part}g =_{df} (f\mathrm{Irr}g) \;\&\; (f\mathrm{Tr}g)$. (f partially orders g).

$f\mathrm{Con}g =_{df} \mathrm{Rel}(f) \;\&\; (Au')(Av')(\langle u' \epsilon g \;\&\; v' \epsilon g \;\&\; \sim T(u'=v')\rangle \langle u',v'\rangle \epsilon f \vee \langle v',u'\rangle \epsilon f)$. (f is a connected relation on g).

$f\mathrm{Tot}g =_{df} (f\mathrm{Irr}g) \;\&\; (f\mathrm{Tr}g) \;\&\; (f\mathrm{Con}g)$. (f totally orders g).

$f\mathrm{We}g =_{df} (f\mathrm{Irr}g) \;\&\; (Ah)(h \subseteq g \;\&\; \sim h=0 \supset (Sx')(x' \epsilon h \;\&\; (Ay')(y' \epsilon h \;\&\; \sim T(y'=x') \supset. \langle x',y'\rangle \epsilon f \;\&\; \sim\langle y',x'\rangle \epsilon f)))$. (f well-orders g).

T.65. $f\mathrm{We}g \supset f\mathrm{Tot}g$.

T.66. $f\mathrm{We}g \;\&\; h \subseteq g \supset f\mathrm{We}h$.

$E =_{df} \{\langle x',y'\rangle \,/\, T(x' \epsilon y')\}$. (The membership relation).

$\mathrm{Trans}_1\,(f) =_{df} (Ag')(g' \epsilon f \supset g' \subseteq f)$. (f is transitive over sets).

$\mathrm{Trans}_2\,(f) =_{df} (Ak)(A\ell)(k \leqslant \ell \;\&\; \ell \epsilon f \supset k \epsilon f)$. (f is transitive over individuals).

$\mathrm{Sect}_g\,(f,h) =_{df} h \subseteq f \;\&\; (Au')(Av')(u' \epsilon f \;\&\; v' \epsilon h \;\&\; \langle u',v'\rangle \epsilon g \supset u' \epsilon h)$. (h is a g-section of f).

$\mathrm{Seg}_g\,(f_1 y') =_{df} \{x' \,/\, x' \epsilon f \;\&\; \langle x',y'\rangle \epsilon g\}$. (The g-segment of f determined by y').

T.67. $\mathrm{Trans}_1\,(f) \equiv. U(f) \subseteq f$.

T.68. $\mathrm{Trans}_1\,(f) \;\&\; \mathrm{Trans}_1\,(g) \supset \mathrm{Trans}_1\,(f \cup g) \;\&\; \mathrm{Trans}_1\,(f \cap g)$.

T.69. (1) $Seg_E\ (f,g')=\{x\ /\ x\epsilon f\ \&\ x\epsilon g'\}=f\cap g'$.

(2) $Seg_E\ (f,k)=0$.

T.70. $M(Seg_E(f,g'))$.

T.71. $Trans_1\ (f) \equiv_{df} (Ag')(g'\epsilon f \supset Seg_E\ (f,g')=g')$.

T.72. $\underline{EWef\ \&\ Sect_E\ (f,h)\ \&\ {\sim}h=f \supset (Su')(u'\epsilon f\ \&\ h=Seg_E\ (f,u'))}$.

Hyp : $EWef\ \&\ Sect_E\ (f,h)\ \&\ {\sim}h=f$ ____(1)

(1), Defns. $Sect_E$: $R \subseteq f\ \&\ (Au')(Av')(u'\epsilon f\ \&\ v'\epsilon h\ \&\ T(u'\epsilon v') \supset u'\epsilon h)$ ____(2)

(1), Defns. We,E : $(EIrrf)\ \&\ (Ah)(h \subseteq f\ \&\ {\sim}h=0 \supset (Sy')(y'\epsilon h\ \&\ (Av')$
$(v'\epsilon h\ \&\ {\sim}T(v'=y') \supset T(y'\epsilon v')\ \&\ {\sim}T(v'\epsilon y'))))$ ____(3)

(1), (2) : ${\sim}f\text{-}h=0$ ____(4)

(2), (3), (4) : $(Su')(u'\epsilon f\text{-}h\ \&\ (Av')(v'\epsilon f\text{-}h\ \&\ {\sim}T(v'=u') \supset T(u'\epsilon v')\ \&$
${\sim}T(v'\epsilon u')))$ ____(5)

Hyp : $w'\epsilon h$ ____(6)

(5) : $u_1'\epsilon f$ ____(7) (u_1' is a constant instantiating the u' of (4)).

(1), (6), (7), T.65 : $T(w'\epsilon u_1')\ v\ T(u_1'\epsilon w')$ ____(8)

Hyp : $T(u_1'\epsilon w')$ ____(9)

(7), (6), (9), (2) : $u_1'\epsilon h$ ____(10)

(5) : ${\sim}u_1'\epsilon h$ ____(11)

(8), (9),(10), (11) : $T(w'\epsilon u_1')$ ____(12)

(2), (6), (12) : $w'\epsilon h \supset w'\epsilon f\ \&\ T(w'\epsilon u_1')$ ____(13)

Hyp : $w' \epsilon f$ & $T(w' \epsilon u_1')$ ____(14)

Hyp : $\sim w' \epsilon h$ ____(15)

(5), (14), (15) : $\sim T(w' = u_1') \supset T(u_1' \epsilon w')$ & $\sim T(w' \epsilon u_1')$____(16)

(14), (16) : $T(w' = u_1')$____(17)

(14), (17) : $T(u_1' \epsilon u_1')$ ____(18)

(1), Defn. We : $\sim T(u_1' \epsilon u_1')$ ____(19)

(15),(18), (19) : $w' \epsilon h$ ____(20)

(14), (20) : $w' \epsilon f$ & $T(w' \epsilon u_1') \supset w' \epsilon h$ ____(21)

(13), (21) : $w' \epsilon h \equiv w' \epsilon f$ & $T(w' \epsilon u_1')$ ____(22)

(7), (22), Defn. Seg : $(Su')(u' \epsilon f$ & $h = Seg_E\ (f, u'))$ ____(23)

(1), (23) : T.72.

Ord (f) $=_{df}$ EWef & $Trans_1$ (f) & $\sim(Sk)(k \epsilon f)$. (f is an ordinal).

Note the restriction here on the definition of an ordinal. This is necessary to prevent each individual from generating a sequence, {k}, {k,{k}}, {k,{k}, {k,{k}}}, etc., which would satisfy the definition of ordinals but would not satisfy the uniqueness requirement.

Ord (k) $=_{df} \sim(kok)$.

On $=_{df}$ {x' / Ord (x')}. (The class of all ordinal numbers).

T.73. $0 \epsilon On$.

Use Greek letters for variables over ordinal numbers.

T.74. $\mathrm{Ord}\ (f) \supset . \sim f \in f \ \& \ (Ag')(g' \in f \supset \sim g' \in g')$.

T.75. $\underline{\mathrm{Ord}\ (f) \ \& \ g \subset f \ \& \ \mathrm{Trans}_1\ (g) \supset g \in f}$.

Hyp : $\mathrm{Ord}\ (f) \ \& \ g \subset f \ \& \ \mathrm{Trans}_1\ (g)$ ____(1)

Hyp : $\mathfrak{u}' \in f \ \& \ v' \in g \ \& \ T(\mathfrak{u}' \in v')$ ____(2)

(2) : $\mathrm{Cl}\ (v')$ ____(3)

(1), (2), (3) : $\mathfrak{u}' \in g$ ____(4)

(2), (4), Def. Sect : $\mathrm{Sect}_E\ (f,g)$ ____(5)

(1), (5), T.72 : $(S\mathfrak{u}')(\mathfrak{u}' \in f \ \& \ g = \mathrm{Seg}_E\ (f,\mathfrak{u}'))$ ____(6)

(6) : $\mathfrak{u}_1' \in f \ \& \ g = \mathrm{Seg}_E\ (f,\mathfrak{u}_1')$ ____(7) ($\mathfrak{u}_1'$ is a constant).

(1), (7), Defn. Ord : $\mathrm{Cl}\ (\mathfrak{u}_1')$ ____(8)

(7), (8), T.69 : $g = f \cap \mathfrak{u}_1'$ ____(9)

(1), (7), (8), Defn. Ord : $\mathfrak{u}_1' \subseteq f$ ____(10)

(9), (10) : $g = \mathfrak{u}_1'$ ____(11)

(7), (11) : $g \in f$ ____(12)

(1), (12) : T.75.

T.76. $\mathrm{Ord}\ (f) \ \& \ \mathrm{Ord}\ (g) \supset . g \subset f \equiv g \in f$.

T.77. $\mathrm{Ord}\ (f) \ \& \ \mathrm{Ord}\ (g) \supset . (g \in f \vee g = f \vee f \in g) \ \& \sim (f \in g \ \& \ g \in f) \ \& \sim (f \in g \ \& \ f = g)$.

T.78. $\underline{\mathrm{Ord}\ (f) \ \& \ g \in f \supset g \in \mathrm{On}}$.

Hyp : $\mathrm{Ord}\ (f) \ \& \ g \in f$ ____(1)

(1), T.74 : $g \subset f$ ____(2)

(1), (2), T.66 : $EWe g$ ____(3)

Hyp : $u \epsilon g$ & $v \epsilon u$ ____(4)

(1), (4), Defn. Ord : $u \epsilon f$ & $v \epsilon f$ & Cl (u) & Cl (v) ____(5)

(1), (5), Defn. Ord : $v \epsilon g$ v $v = g$ v $g \epsilon v$ ____(6)

(1), T.74 : $\sim v = g$ & $\sim g \epsilon v$ ____(7)

(6), (7) : $v \epsilon g$ ____(8)

(4), (8) : Trans_1 (g) ____(9)

(2), Defn. Ord : $(Ak) \sim (k \epsilon g)$ ____(10)

(1), (3), (9), (10) : T.78.

T.79. EWeOn.

T.80. Ord (On).

T.79 : EWeOn ____(1)

T.78 : Trans_1 (On) ____(2)

Defn. Ord : $(Ak) \sim (k \epsilon On)$ ____(3)

(1), (2), (3), Defn. Ord : Ord (On).

T.81. Pr (On).

T.82. Ord (f) $\supset$. $f = On$ v $f \epsilon On$.

$f <_o g =_{df} f \epsilon On$ & $g \epsilon On$ & $f \epsilon g$.

$f \leqslant_o g =_{df} g \epsilon On$ & $(f = g$ v $f <_o g)$.

T.83. $(A\beta)((A\alpha)(\alpha \epsilon \beta \supset \alpha \epsilon f) \supset \beta \epsilon f) \supset On \subseteq f$. T.83 can be used to prove that all ordinals have a given property $\phi(\alpha)$, provided the universal closure of $\phi(\alpha)$ is significant. Let $f = \{g' / \phi(g')$ & $g' \epsilon On\}$. Then show that $(A\beta)((A\alpha)(\alpha \epsilon \beta \supset \phi(\alpha)) \supset \phi(\beta)$.

$(f')' =_{df} f' \cup \{f'\}$.

T.84. <u>$f' \in On \equiv (f')' \in On$.</u>

If $f' \in On$, then one also needs to prove that $(Ak) \sim (k \in (f')')$, i.e. $(Ak) \sim (k \in f' \cup \{f'\})$. This follows from $(Ak) \sim (k \in f')$.

T.85. $(A\alpha) \sim (S\beta)(\alpha <_o \beta <_o \alpha')$.

T.86. $(A\alpha)(A\beta)(\alpha' = \beta' \supset \alpha = \beta)$.

Suc $(f') =_{df} f' \in On$ & $(S\alpha)(f' = \alpha')$. (f' is a successor ordinal).

Suc $(k) =_{df} \sim(kok)$.

$K_I =_{df} \{x' \; / \; x' = 0 \vee \text{Suc }(x')\}$. ($K_I$ is the class of ordinals of the first kind).

$\omega =_{df} \{x' \; / \; x' \in K_I$ & $T(Ag')(g' \in x' \supset g' \in K_I)\}$.

$1 =_{df} \{0\}$. (the number one).

T.87. $0 \in \omega$ & $1 \in \omega$.

T.88. $M(\omega)$.

T.89. $(A\alpha)(\alpha \in \omega \equiv \alpha' \in \omega)$.

T.90. $0 \in f$ & $(Ag')(g' \in f \supset (g')' \in f) \supset \omega \subseteq f$.

T.91. $(A\alpha)(\alpha \in \omega$ & $\beta <_o \alpha \supset \beta \in \omega)$.

The members of ω are called finite ordinals.

$2 =_{df} 1'$, $3 =_{df} 2'$, ..., $n+1 =_{df} n'$, ... Hence $0 \in \omega$, $1 \in \omega$, $2 \in \omega$, etc.

Lim $(f') =_{df} f' \in On$ & $\sim f' \in K_I$. (f' is a limit ordinal).

T.92. Lim (ω).

T.93. $(Af)(f \subseteq On \supset (U(f)\epsilon On \ \& \ (A\alpha)(\alpha\epsilon f \supset \alpha \leqslant_o U(f)) \ \& \ (A\beta)((A\alpha)(\alpha\epsilon f \supset \alpha\leqslant_o\beta) \supset U(f)\leqslant_o\beta)))$. (If f is a set of ordinals then U(f) is an ordinal which is the least upper bound of f).

$(Ak)\sim(k\epsilon U(f))$ is clear since U(f) is a set of ordinals.

T.94. $(Af)(f \subseteq On \ \& \ \sim f=0 \ \& \ (A\alpha)(\alpha\epsilon f \supset (S\beta)(\beta\epsilon f \ \& \ \alpha<_o\beta)) \supset Lim\ (U(f))$.

(If f is a non-empty set of ordinals without a maximum, then U(f) is a limit ordinal).

T.95. $(A\alpha)(Suc(\alpha) \supset ((U(\alpha))'=\alpha) \ \& \ (Lim\ (\alpha) \supset U(\alpha)=\alpha))$.

T.96. $0\epsilon f \ \& \ (A\alpha)(\alpha\epsilon f \supset \alpha'\epsilon f) \ \& \ (A\alpha)(Lim\ (\alpha) \ \& \ (A\beta)(\beta<_o\alpha \supset \beta\epsilon f) \supset \alpha\epsilon f) \supset On \subseteq f$. (Transfinite Induction).

T.97. $0\epsilon f \ \& \ (A\alpha)(\alpha'<_o\delta \ \& \ \alpha\epsilon f \supset \alpha'\epsilon f) \ \& \ (A\alpha)(\alpha<_o\delta \ \& \ Lim\ (\alpha) \ \& \ (A\beta)(\beta<_o\alpha \supset \beta\epsilon f) \supset \alpha\epsilon f) \supset \delta \subseteq f$. (Transfinite Induction up to δ).

T.98. $(Af)(S!g)(Fnc(g) \ \& \ \mathfrak{D}(g)=On \ \& \ (A\alpha)(g`\alpha=f`(\alpha 1 g)))$.

In the proof, let $Y_1=\{u' \ / \ (Sf')(T(f'=u') \ \& \ Fnc\ (f') \ \& \ ......)\}$.

T.99. $(Af')(Af_1)(Af_2)(S!g)(Fnc(g) \ \& \ \mathfrak{D}(g)=On \ \& \ g`0=f' \ \& \ (A\alpha)(g`(\alpha')=f_1`(g`\alpha)) \ \& \ (A\alpha)(Lim\ (\alpha) \supset g`\alpha=f_2`(\alpha 1 g)))$. (T.98 and T.99 provide definitions of unique functions by transfinite induction).

T.100. $(Af')(Af_1)(Af_2)(S!g)(Fnc(g) \ \& \ \mathfrak{D}(g)=\delta \ \& \ g`0=f' \ \& \ (A\alpha)(\alpha'<_o\delta \supset g`(\alpha')=f_1`(g`\alpha)) \ \& \ (A\alpha)(Lim\ (\alpha) \ \& \ \alpha<_o\delta \supset g`\alpha=f_2`(\alpha 1 g)))$.

(The definition of a unique function for ordinals up to δ).

$f \underset{h}{\simeq} g =_{df} Fnc\ (h) \ \& \ Un_1\ (h) \ \& \ \mathfrak{D}(h)=f \ \& \ \mathfrak{R}(h)=g$.

$f \simeq g =_{df} (Sh)(f \underset{h}{\simeq} g)$. (f and g are equinumerous).

T.101. $f' \simeq g' \equiv (Sh')(f' \underset{h}{\simeq} g')$.

T.102. $f \underset{h}{\simeq} g \supset g \underset{\breve{h}}{\simeq} f$.

h is the composition of f and g iff $h=\{\langle x',y'\rangle \;/\; (Sz')(\langle x',z'\rangle \in f \;\&\; \langle z',y'\rangle \in g)\}$.

T.103. $f \underset{h_1}{\simeq} g \;\&\; g \underset{h_2}{\simeq} h \supset f \underset{h_3}{\simeq} h$, where h_3 is the composition of h_1 and h_2.

T.104. $f \simeq f$.

T.105. $f \simeq g \supset g \simeq f$.

T.106. $f \simeq g \;\&\; g \simeq h \supset f \simeq h$.

T.107. $f \simeq g \;\&\; f_1 \simeq g_1 \;\&\; f \cap f_1 = 0 \;\&\; g \cap g_1 = 0 \supset f \cup f_1 \simeq g \cup g_1$.

T.108. $f \simeq g \;\&\; f_1 \simeq g_1 \supset f \mathrm{x} f_1 \simeq g \mathrm{x} g_1$.

In the proof, let $W=\{\langle u',v'\rangle \;/\; (Sx')(Sy')(x' \in f \;\&\; y' \in f_1 \;\&\; T(u'=\langle x',y'\rangle) \;\&\; T(v'=\langle F`x', G`y'\rangle))\}$.

T.109. $f \mathrm{x} \{g'\} \simeq f$.

In the proof, let $F=\{\langle u',v'\rangle \;/\; u' \in f \;\&\; T(v'=\langle u',g'\rangle)\}$.

T.110. $f \mathrm{x} g \simeq g \mathrm{x} f$.

Similarly, cover the $\ulcorner = \urcorner$ with a $\ulcorner T \urcorner$.

T.111. $(f \mathrm{x} g) \mathrm{x} h \simeq f \mathrm{x} (g \mathrm{x} h)$.

Similarly, cover the $\ulcorner = \urcorner$ with a $\ulcorner T \urcorner$.

T.112. $(Af)(Ag)(Sf_1)(Sg_1)(f \simeq f_1 \;\&\; g \simeq g_1 \;\&\; f_1 \cap g_1 = 0)$.

$f^g =_{df} \{u' \;/\; (Sf')(T(f'=u') \;\&\; \mathrm{Fnc}\,(f') \;\&\; \mathcal{D}(f')=g \;\&\; \mathcal{R}(f') \subseteq f)\}$.

(The class of all sets which are functions from g into f).

T.113. $\mathcal{P}(f) \simeq 2^f$.

T.114. $\mathrm{Pr}(g) \supset f^g = 0$.

T.115. $M(f'^{g'})$.

T.116. $f^0=\{0\}=1$.

T.117. $\sim f=0 \supset 0^f=0$.

T.118. $f \simeq g \;\&\; h \simeq h_1 \supset f^h \simeq g^{h_1}$.

T.119. $f \cap g=0 \supset h^{f \cup g} \simeq h^f \times h^g$.

T.120. $(f^g)^h \simeq f^{g \times h}$.

$f \preceq g =_{df} (Sh)(h \subseteq g \;\&\; f \simeq h)$. (f is equinumerous with a subclass of ~~h~~ g).

$f \prec g =_{df} f \preceq g \;\&\; \sim(f \simeq g)$.

T.121. $f \preceq g \equiv f \prec g \vee f \simeq g$.

T.122. $f \preceq g \;\&\; Pr(f) \supset Pr(g)$.

T.123. $f \preceq f \;\&\; \sim(f \prec f)$.

T.124. $f \subseteq g \supset f \preceq g$.

T.125. $f \preceq g \;\&\; g \preceq h \supset f \preceq h$.

T.126. $f \preceq g \;\&\; g \preceq f \supset f \simeq g$.

T.127. $f \preceq g \;\&\; f_1 \preceq g_1 \supset. (g \cap g_1=0 \supset f \cup f_1 \preceq g \cup g_1) \;\&\; f \times f_1 \preceq g \times g_1 \;\&\; (M(g_1) \supset f^{f_1} \preceq g^{g_1})$.

T.128. $f \preceq f \cup g$.

T.129. $f \prec g \supset \sim(g \preceq f)$.

T.130. $f \prec g \;\&\; g \preceq h \supset f \prec h$.

T.131. $f' \prec \mathcal{P}(f')$.

Fin (f) $=_{df} (S\alpha)(\alpha \in \omega \;\&\; f \simeq \alpha)$. (f is finite).

Den (f) $=_{df} f \simeq \omega$. (f is denumerable).

Inf (f) $=_{df} \sim$Fin (f). (f is infinite).

The standard results about finite, infinite and denumerable classes now follow as in Mendelson.

T.132. For any set f', there is an ordinal which is not equinumerous with any subset of x.

In the proof, let the relation R be such that $<u',v'>\in$ RiffT(f'' $u'\in f'$' v'). Define an <u>initial ordinal</u> as an ordinal which is not equinumerous with any smaller ordinal. It follows that every ordinal, α, is equinumerous with a unique initial ordinal, $\omega_\beta \leqslant_o \alpha$, namely, with the least ordinal equinumerous with α. So the initial ordinals can be regarded as the cardinal numbers. Now there follows the standard theory of initial ordinals.

This development is now sufficient to deal with the Axioms of Constructibility and Choice and the Generalised Continuum Hypothesis.

<u>Further Special Class Axioms.</u>

A.C. $(Af')(f'\epsilon g' \supset \sim f'=0 \ \& \ (Ah')(h'\epsilon g' \ \& \sim h'=f' \supset h'\cap f'=0)) \supset (Sj')$
$(Af')(f'\epsilon g' \supset (S!x')(x'\epsilon f'\cap j'))$.

D. $(Af)((Sg')(g'\epsilon f) \supset (Sg')(g'\epsilon f \ \& \sim(Sh')(h'\epsilon g' \ \& \ h'\epsilon f)))$.

GCH. $(Af')\sim(Sg')(f' \prec g' \prec \mathcal{P}(f'))$.

C. f' is constructible (to be defined later).

As in Mendelson, it can be shown that AC is equivalent to each of the following:

(1) For any set f', there is a function g' such that, for any non-empty subset h' of f', g''$h'\epsilon h'$.

(*): I is a set by Individual Axiom 3. By the 1-1 correspondence between I and M_o, using Axiom R, M_o is also a set.

(2) Every set can be well-ordered.

(3) $f' \preceq g' \vee g' \preceq f'$.

(4) Any non-empty partially-ordered set f', in which every chain (i.e. totally-ordered subset) has an upper bound, has a maximal element.

Notice the difference between my Axiom D and Mendelson's Axiom of Restriction. Individuals may belong to the intersection of g' and f. Now I will define the notion of constructible set, which is similar to that on p.87 of [3]. Define the set M_o as follows: $\mathfrak{u}'\epsilon M_o \equiv (Sk)T(\mathfrak{u}'=\{k\})$. M_o is a set by Individual Axiom 3 and Axiom R.* If α is a limit ordinal, then the set M_α is defined as the union of all the sets M_β, for $\beta <_o \alpha$, i.e. $\mathfrak{u}'\epsilon M_\alpha \equiv (S\beta)(\beta<_o\alpha \; \& \; \mathfrak{u}'\epsilon M_\beta)$. The set $M_{\alpha+1}$ is defined as the union of the set M_α and the set of all sets f' for which there is a formula $A(z',w_1',\ldots,w_\ell')$, which is significant for all substitutions into its free variables, such that if $A_{M_\alpha \cup I}$ denotes A with all bound variables restricted to $M_\alpha \cup I$, where I is the set of all individuals, then for some (constant) $\overline{w}_i'$ in $M_\alpha \cup I$, for all i, $f'=\{z'\epsilon M_\alpha\cup I \,/\, A_{M_\alpha\cup I}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}$. $[\{z'\epsilon M_\alpha\cup I \,/\, A_{M_\alpha\cup I}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}=\{z' \,/\, z'\epsilon M_\alpha\cup I \; \& \; A_{M_\alpha\cup I}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}]$.

Now we show that $M_{\alpha+1}$ can be defined in the formal system, given that M_α can be so defined. This proof follows that of Cohen's on p.92 of [3]. For each $\mathfrak{r} \geq o$ let $f_{\mathfrak{r}}'$ denote the set of all ordered triples $\langle h_1',h_2',h_3'\rangle$ where h_1', h_2' and h_3' are sets of ordered n-tuples $\langle x_1',\ldots,x_n'\rangle$ for which there is a formula $A(x_1',\ldots,x_n',t_1',\ldots,t_m')$, with exactly r quantifiers, but where A can be non-significant for some substitutions into its free variables, such that $h_1'=\{\langle x_1',\ldots,x_n'\rangle\in(M_\alpha\cup I)^n \,/\, TA_{M_\alpha\cup I}(x_1',\ldots,x_n',\overline{t}_1',\ldots,\overline{t}_m')\}$, $h_2'=\{\langle x_1',\ldots,x_n'\rangle\in(M_\alpha\cup I)^n \,/\, FA_{M_\alpha\cup I}(x_1',\ldots,x_n',\overline{t}_1',\ldots,\overline{t}_m')\}$, and $h_3'=\{\langle x_1',\ldots,x_n'\rangle\in(M_\alpha\cup I)^n \,/\, \sim SA_{M_\alpha\cup I}(x_1',\ldots,x_n',\overline{t}_1',\ldots,\overline{t}_m')\}$, where $\overline{t}_i'\in M_\alpha\cup I$, for all i.

We show that f_r' is expressible in the formal system by an induction on r. Firstly, in order to define f_0', we define g_0' as follows:

$\mathfrak{u}'\in g_{0,n}' \equiv (Sy')(y'\in M_\alpha\cup I \;\&\; (Sh_1')(Sh_2')(Sh_3')(\mathfrak{u}'=\langle h_1',h_2',h_3'\rangle \;\&\; (Aw')(w'\in h_1' \equiv (Sx_1')(Sx_2')(T(w'=\langle x_1',x_2'\rangle) \;\&\; x_1'\in M_\alpha\cup I \;\&\; x_2'\in M_\alpha\cup I \;\&\; T(x_1'\in y'))) \;\&\; (Aw')(w'\in h_2' \equiv (Sx_1')(Sx_2')(T(w'=\langle x_1',x_2'\rangle) \;\&\; x_1'\in M_\alpha\cup I \;\&\; x_2'\in M_\alpha\cup I \;\&\; F(x_1'\in y'))) \;\&\; (Aw')(w'\in h_3' \equiv (Sx_1')(Sx_2')(T(w'=\langle x_1',x_2'\rangle) \;\&\; x_1'\in M_\alpha\cup I \;\&\; x_2'\in M_\alpha\cup I \;\&\; \sim S(x_1'\in y'))))) \vee \ldots\ldots$ (for other types of formulae using ordered tuples from 1 to n).

This example is for the formula $x_1'\in\overline{y}'$ with ordered pairs $\langle x_1',x_2'\rangle$, this particular disjunct yielding a set because of the Axiom R and the assumption that M_α is a set. $g_{0,n}'$ will be a set because it is a finite union of sets. Define g_0' as the union of all the $g_{0,n}'$'s where $n\in\omega-\{0\}$. So g_0' is the set of all ordered triples $\langle h_1',h_2',h_3'\rangle$ where h_1', h_2' and h_3' are sets of ordered n-tuples $\langle x_1',\ldots,x_n'\rangle$ for which there is a formula $A(x_1',\ldots,x_n',t_1',\ldots,t_m')$ with no connectives or quantifiers and such that h_1', h_2' and h_3' are defined as above.

Using an induction on the length of formulae without quantifiers assume,

for all $k<\ell$, the set g_k' has been constructed to deal with all formulae without quantifiers and with k connectives. To construct g_ℓ' we need ordered triples corresponding to formulae with ℓ connectives and obtained from previous formulae by the use of one of $\sim$, $\supset$ and T_n.
$\mathfrak{u}'\epsilon g_\ell' \equiv (Sh_1')(Sh_2')(Sh_3')(<h_1',h_2',h_3'>\epsilon g_{\ell-1}' \,\&\, T(\mathfrak{u}'=<h_2',h_1',h_3'>))$ v $(Sk_1)(Sk_2)(k_1+k_2=\ell-1 \,\&\, (Sh_1')(Sh_2')(Sh_3')(Sh_4')(Sh_5')(Sh_6')(<h_1', h_2',h_3'>\epsilon g_{k_1}' \,\&\, <h_4',h_5',h_6'>\epsilon g_{k_2}' \,\&\, T(\mathfrak{u}'=<(h_1'\cap h_4')\cup\overline{h}_1',h_1'\cap h_5', h_1'\cap h_6'>)))$ v $(Sh_1')(Sh_2')(Sh_3')(<h_1',h_2',h_3'>\epsilon g_{\ell-1}' \,\&\, T(\mathfrak{u}'=<h_1',0, h_2'\cup h_3'>))$, where complements are taken with respect to $(M_\alpha \cup I)^n$ for n-tuples. g_ℓ' is a set because the g_k''s for $k<\ell$ are sets and the h_i''s are sets. Define f_0' as the union of all g_ℓ''s such that $\ell\epsilon\omega$. Now by induction on r we will define f_r'. A set $<h_1',h_2',h_3'>$ will be a member of f_r' <u>either</u> if there is a set $<h_4',h_5',h_6'>\epsilon f_{r-1}'$ such that h_4',h_5' and h_6' are sets of (n+1) - tuples and such that $<x_1',\ldots, x_n'>\epsilon h_1' \equiv (Sx_0')(x_0'\epsilon M_\alpha\cup I \,\&\, <x_0',x_1',\ldots,x_n'>\epsilon h_4')$, $<x_1',\ldots,x_n'>\epsilon h_2' \equiv (Sx_0')(x_0'\epsilon M_\alpha\cup I \,\&\, <x_0',x_1',\ldots,x_n'>\epsilon h_5') \,\&\, \sim(Sx_0')(x_0'\epsilon M_\alpha\cup I \,\&\, <x_0',x_1',\ldots,x_n'>\epsilon h_4')$ and $<x_1',\ldots,x_n'>\epsilon h_3' \equiv (Ax_0')(x_0'\epsilon M_\alpha\cup I \supset <x_0', x_1',\ldots,x_n'>\epsilon h_6')$ <u>or</u> if there is a set $<h_4',h_5',h_6'>\epsilon f_{r-1}'$ such that h_4', h_5' and h_6' are sets of (n+1) - tuples and such that $<x_1',\ldots, x_n'>\epsilon h_1' \equiv (Ax_0')(x_0'\epsilon M_\alpha\cup I \supset <x_0',x_1',\ldots,x_n'>\epsilon h_4')$, $<x_1',\ldots,x_n'>\epsilon h_2' \equiv (Sx_0')(x_0'\epsilon M_\alpha\cup I \,\&\, <x_0',\ldots,x_n'>\epsilon h_5' \,\&\, \sim(Sx_0')(x_0'\epsilon M_\alpha\cup I \,\&\, <x_0', \ldots,x_n'>\epsilon h_6')$ and $<x_1',\ldots,x_n'>\epsilon h_3' \equiv (Sx_0')(x_0'\epsilon M_\alpha\cup I \,\&\, <x_0',\ldots,x_n'> \epsilon h_6')$.

Then the set $M_{\alpha+1}$ is defined as the union of the set M_α with the set of all sets h_1', where, in the ordered triple $\langle h_1', h_2', h_3' \rangle$ which belongs to some f_r', h_3' is the null set and h_1' and h_2' are sets of 1-tuples.

Thus the Axiom of Constructibility (Axiom C), in the form $(Af')(S\alpha)(f' \epsilon M_\alpha)$, can be formally defined in the system.

We now prove a theorem showing that only the connectives $\sim$, & and T and quantifier A need be used in the predicate ϕ of the Axiom B to generate all the classes that Axiom B generates.

Theorem 1.

If ϕ is significant for all substitutions into its free variables, then there is a ϕ' such that $\phi \equiv \phi'$ and ϕ' contains only the connectives $\sim$, & and T and the quantifier A.

Proof.

This proof is similar to the proof that $M_{\alpha+1}$ can be defined formally, given M_α.

There are finitely many atomic formulae occurring in ϕ. Corresponding to each one there are three classes defined as follows:

If the atomic formula is $x' \epsilon \overline{y}$, say, then $(Ax')(x' \epsilon h_1 \equiv T(x' \epsilon \overline{y}))$, $(Ax')(x' \epsilon h_2 \equiv F(x' \epsilon \overline{y}))$ and $(Ax')(x' \epsilon h_3 \equiv \sim S(x' \epsilon \overline{y}))$ give definitions of the three classes, h_1, h_2 and h_3. If the atomic formula is $x_1' \epsilon x_2'$, say, then $(Ax_1')(Ax_2')(\langle x_1', x_2' \rangle \epsilon h_1 \equiv T(x_1' \epsilon x_2'))$, $(Ax_1')(Ax_2')(\langle x_1', x_2' \rangle \epsilon h_2 \equiv F(x_1' \epsilon x_2'))$ and $(Ax_1')(Ax_2')(\langle x_1', x_2' \rangle \epsilon h_3 \equiv \sim S(x_1' \epsilon x_2'))$ give definitions of h_1, h_2 and h_3. And so on for any atomic formula appearing in ϕ. If the atomic formula has none or one free variable then the h's have 1-tuples for members and if the atomic formula has two free variables then the h's have 2-tuples for members.

We now assume that h_1, h_2 and h_3 have been found for any predicate ϕ with less than n connectives and quantifiers and take the quantifiers and connectives in turn.

Let h_1, h_2 and h_3 be the classes for ϕ and form $\sim\phi$. $(Ax_1',\ldots,x_m')(\langle x_1',\ldots,x_m'\rangle\epsilon h_4 \equiv \langle x_1',\ldots,x_m'\rangle\epsilon h_2)$, $(Ax_1',\ldots,x_m')(\langle x_1',\ldots,x_m'\rangle\epsilon h_5 \equiv \langle x_1',\ldots,x_m'\rangle\epsilon h_1)$ and $(Ax_1',\ldots,x_m')(\langle x_1',\ldots,x_m'\rangle\epsilon h_6 \equiv \langle x_1',\ldots,x_m'\rangle\epsilon h_3)$ define the classes h_4, h_5 and h_6 for $\sim\phi$.

Let h_1, h_2 and h_3 be the classes for ϕ_1 (where h_1, h_2 and h_3 have members of the form $\langle x_{i_1}',\ldots,x_{i_k}'\rangle$) and let h_4, h_5 and h_6 be the classes for ϕ_2 (where h_4, h_5 and h_6 have members of the form $\langle x_{j_1}',\ldots,x_{j_\ell}'\rangle$). $(Ax_{i_1}',\ldots,x_{j_\ell}')(\langle x_{i_1}',\ldots,x_{j_\ell}'\rangle\epsilon h_7 \equiv (\langle x_{i_1}',\ldots,x_{i_k}'\rangle\epsilon h_1 \,\&\, \langle x_{j_1}',\ldots,x_{j_\ell}'\rangle\epsilon h_4) \vee \sim\langle x_{i_1}',\ldots,x_{i_k}'\rangle\epsilon h_1)$, $(Ax_{i_1}',\ldots,x_{j_\ell}')(\langle x_{i_1}',\ldots,x_{j_\ell}'\rangle\epsilon h_8 \equiv \langle x_{i_1}',\ldots,x_{i_k}'\rangle\epsilon h_1 \,\&\, \langle x_{j_1}',\ldots,x_{j_\ell}'\rangle\epsilon h_5)$ and $(Ax_{i_1}',\ldots,x_{j_\ell}')(\langle x_{i_1}',\ldots,x_{j_\ell}'\rangle\epsilon h_9 \equiv \langle x_{i_1}',\ldots,x_{i_k}'\rangle\epsilon h_1 \,\&\, \langle x_{j_1}',\ldots,x_{j_\ell}'\rangle\epsilon h_6)$ define the classes h_7, h_8 and h_9 for $\phi_1 \supset \phi_2$, where $x_{i_1}',\ldots,x_{j_\ell}'$ contains no repetition of variables.

Let h_1, h_2 and h_3 be the classes for ϕ and form $T_n\phi$. $(Ax_1',\ldots,x_p')(\langle x_1',\ldots,x_p'\rangle\epsilon h_4 \equiv \langle x_1',\ldots,x_p'\rangle\epsilon h_1)$, $(Ax_1',\ldots,x_p')(\langle x_1',\ldots,x_p'\rangle\epsilon h_5 \equiv 0\epsilon 0)$ and $(Ax_1',\ldots,x_p')(\langle x_1',\ldots,x_p'\rangle\epsilon h_6 \equiv \sim\langle x_1',\ldots,x_p'\rangle\epsilon h_1)$ define the classes h_4, h_5 and h_6 for $T_n\phi$.

Let h_1, h_2 and h_3 be the classes for $\phi(x')$ and form $(Ax')\phi(x')$. $(Ax_1',\ldots,x_k')(\langle x_1',\ldots,x_k'\rangle\epsilon h_4 \equiv (Ax')(\langle x_1',\ldots,x',\ldots,x_k'\rangle\epsilon h_1))$, $(Ax_1',\ldots,x_k')(\langle x_1',\ldots,x_k'\rangle\epsilon h_5 \equiv (Sx')(\langle x_1',\ldots,x',\ldots,x_k'\rangle\epsilon h_2)$ & $\sim(Sx')(\langle x_1',\ldots,x',\ldots,x_k'\rangle\epsilon h_3))$ and $(Ax_1',\ldots,x_k')(\langle x_1',\ldots,x_k'\rangle\epsilon h_6 \equiv (Sx')(\langle x_1',\ldots,x',\ldots,x_k'\rangle\epsilon h_3))$ define the classes h_4, h_5 and h_6 for $(Ax')\phi(x')$.

$(Ax_1',\ldots,x_k')(\langle x_1',\ldots,x_k'\rangle\epsilon h_4 \equiv (Sx')(\langle x_1',\ldots,x',\ldots,x_k\rangle\epsilon h_1))$, $(Ax_1',\ldots,x_k')(\langle x_1',\ldots,x_k'\rangle\epsilon h_5 \equiv (Sx')(\langle x_1',\ldots,x',\ldots,x_k'\rangle\epsilon h_2)$ & $\sim(Sx')(\langle x_1',\ldots,x',\ldots,x_k'\rangle\epsilon h_1))$ and $(Ax_1',\ldots,x_k')(\langle x_1',\ldots,x_k'\rangle\epsilon h_6 \equiv (Ax')(\langle x_1',\ldots,x',\ldots,x_k'\rangle\epsilon h_3))$ define the classes h_4, h_5 and h_6 for $(Sx')\phi(x')$.

Hence, for any formula ϕ there are corresponding classes h_1, h_2 and h_3 such that $\langle x_1',\ldots,x_n'\rangle\epsilon h_1 \equiv T\phi$, $\langle x_1',\ldots,x_n'\rangle\epsilon h_2 \equiv F\phi$ and $\langle x_1',\ldots,x_n'\rangle\epsilon h_3 \equiv \sim S\phi$, because of the method of constructing the h's for the formula ϕ. Since ϕ is significant for all substitutions into its free variables, h_3 is the null class. Hence $\langle x_1',\ldots,x_n'\rangle\epsilon h_1 \equiv \phi$, where h_1 was constructed using only $\sim$, &, T and A, the uses of the quantifier S being replaceable by $\sim A\sim$ because S only quantifies two-valued formulae. Hence the ϕ' required can be taken as $\langle x_1',\ldots,x_n'\rangle\epsilon h_1$.

The next task is to prove that the formal theory is relatively consistent to the set theory, Z-F. This is more difficult than would first appear since, in usual set or class theories containing individuals, there is no axiom asserting the existence of at least one individual and hence one can ignore individuals when constructing a model or showing consistency in any way. But in this theory containing Individual Axiom 5 (necessary of course for the development of a theory of individuals) we cannot ignore individuals when constructing a model for the theory. Since the theory of individuals can be shown to be consistent using a model consisting of only one individual, we will construct a model for the special class theory also containing only one individual. The model cannot be an inner model of any class theory because there is no such class theory explicitly containing individuals.

We first construct a model N for the individuals and sets of the theory and then extend it to a model N' for the individuals and special classes of the theory. The domain of N and the valuations of the membership statements are constructed by a transfinite induction on the ordinals. This is similar to the construction of the constructible sets of the inner model of Z-F, that appears in [3], p.87. The final aim is to establish a domain with the following members: $k,\{k\},M_\beta$, for all ordinals β, and all expressions of the form: $\{z'\epsilon M_\alpha \cup \{k\} / A_{M_\alpha \cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}$, where $\overline{w}_i'$ is k or $\overline{w}_i'\epsilon M_\alpha$ has the value 1 in the value assignment to follow, where all the bound variables in $A_{M_\alpha \cup\{k\}}$ are restricted to $M_\alpha \cup\{k\}$, and $A_{M_\alpha \cup\{k\}}$ has the value 1 or 0 for all substitutions into its free variable z'.

The restrictions of variable to $M_\alpha \cup \{k\}$ are done as follows:

$(Ax')(x' \epsilon M_\alpha \supset f(x'))$ & $f(k)$.

$(Sx')(T_n(x' \epsilon M_\alpha)$ & $f(x'))$ v $f(k)$.

Assume that these restrictions to $M_\alpha \cup \{k\}$ apply for the whole construction of the domain N.

The transfinite induction is as follows:

We shall use the notation, v(expression) =1,0 or n. We will construct a transfinite sequence of domains, $D_0 \subseteq D_1 \subseteq D_2 \subseteq \ldots\ldots \subseteq D_\alpha \subseteq D_{\alpha+1} \subseteq \ldots\ldots \subseteq D^U \subseteq D^0 \subseteq D^1 \subseteq \ldots\ldots \subseteq D^h \subseteq \ldots\ldots \subseteq D^S$, where D^U will be the domain for sets and individual and D^S will be the domain for special classes and individual. The valuations made for each domain will hold good for all domains containing it.

Let the domain D_0 consist of k (the individual) and $\{k\}$. Then $v(kok)=1$, $v(\{k\}ok)=v(ko\{k\})=v(\{k\}o\{k\})=n$, $v(k\epsilon k)=v(\{k\}\epsilon k)=n$, $v(k\epsilon\{k\})=1$, $v(\{k\}\epsilon\{k\})=0$.

Let the domain D_1 consist of k, $\{k\}$, M_0, and all expressions of the form:

$\{z' \epsilon M_0 \cup \{k\} \;/\; A_{M_0\cup\{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}$, where $\overline{w}_i' \epsilon D_0$, for all i, and, for all $z' \epsilon D_0$, $A_{M_0\cup\{k\}}$ has the value 1 or 0. If g' and $h' \epsilon D_1 - D_0$, then $v(kog')=v(g'ok)=v(g'oh')=v(\{k\}og')=v(g'o\{k\})=n$. Also $v(k\epsilon M_0)= 0$, $v(\{k\}\epsilon M_0)=1$, $v(M_0\epsilon k)=n$, $v(M_0\epsilon\{k\})=0$, $v(M_0\epsilon M_0)=0$. If $g' \epsilon D_1-(D_0 \cup \{M_0\})$, then $v(M_0\epsilon g')=0$, $v(g'\epsilon k)=n$, $v(g'\epsilon\{k\})=0$.

$v(x' \epsilon \{z' \epsilon M_0\cup\{k\} \;/\; A_{M_0\cup\{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = v(A_{M_0\cup\{k\}}(x', \overline{w}_1', \ldots, \overline{w}_\ell'))$, for all $x' \epsilon D_0$, where the range of bound variables is taken as D_0 for the valuation. If $g' \epsilon D_1 - (D_0 \cup \{M_0\})$, then if $v(x' \epsilon g') = v(x' \epsilon \{k\})$ for all $x' \epsilon D_0$, then $v(g' \epsilon M_0) = 1$. Call $\{k\}$ the corresponding member of D_0 for g'.

If it is not the case that $v(x' \epsilon g') = v(x' \epsilon \{k\})$ for all $x' \epsilon D_0$, then $v(g' \epsilon M_0) = 0$.

Let $f' \epsilon D_1 - (D_0 \cup \{M_0\})$. Let $v(f' \epsilon M_0) = 1$. Then $v(f' \epsilon \{z' \epsilon M_0 \cup \{k\} / B_{M_0 \cup \{k\}}(z', \overline{x}_1', \ldots, \overline{x}_m')\}) = v(B_{M_0 \cup \{k\}}(\{k\}, \overline{x}_1', \ldots, \overline{x}_m'))$, where $\overline{x}_i' \epsilon D_0$ for all i, and the range of bound variables in $B_{M_0 \cup \{k\}}$ is taken as D_0. Now let $v(f' \epsilon M_0) = 0$. Then $v(f' \epsilon \{z' \epsilon M_0 \cup \{k\} / B_{M_0 \cup \{k\}}(z', \overline{x}_1', \ldots, \overline{x}_m')\}) = 0$. This completes the valuation for D_1.

We shall now show that the Axiom of Extensionality holds in D_1.

Let $v(x' \epsilon f') = v(x' \epsilon g')$ for all $x' \epsilon D_1$, where $f', g' \epsilon D_1 - \{k\}$.

$v(f' \epsilon \{k\}) = 0 = v(g' \epsilon \{k\})$.

(i) Let $v(f' \epsilon M_0) = 1$ and $f' \epsilon D_0$. Then f' is $\{k\}$. (a) Let $g' \epsilon D_0$, then g' is $\{k\}$, which is f'. Hence $v(g' \epsilon M_0) = 1$ and $v(f' \epsilon \{z' \epsilon M_0 \cup \{k\} / A_{M_0 \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = v(g' \epsilon \{z' \epsilon M_0 \cup \{k\} / A_{M_0 \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$.

(b) Let $g' \epsilon D_1 - (D_0 \cup \{M_0\})$. Then f' is a corresponding member of D_0 for g', $v(g' \epsilon M_0) = 1$ and $v(g' \epsilon \{z' \epsilon M_0 \cup \{k\} / A_{M_0 \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$ $= v(A_{M_0 \cup \{k\}}(f', \overline{w}_1', \ldots, \overline{w}_\ell')) = v(f' \epsilon \{z' \epsilon M_0 \cup \{k\} / A_{M_0 \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$. (c) Let g' be M_0. This cannot be the case because $v(k \epsilon f') = 1$ and $v(k \epsilon M_0) = 0$.

(ii) Let $v(f'\epsilon M_0) = 1$ and $f'\epsilon D_1 - D_0$. f' cannot be M_0 because $v(M_0\epsilon M_0) = 0$. (a) Let $g'\epsilon D_0$, then this case has already been treated in (i). (b) Let $g'\epsilon D_1 - (D_0\cup\{M_0\})$. f' has a corresponding member, $\{k\}$, of D_0. Hence $v(x'\epsilon f') = v(x'\epsilon\{k\})$ for all $x'\epsilon D_0$, and $v(f'\epsilon\{z'\epsilon M_0\cup\{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}) = v(A_{M_0\cup\{k\}}(\{k\},\overline{w}_1',\ldots,\overline{w}_\ell'))$. $v(x'\epsilon g') = v(x'\epsilon\{k\})$ for all $x'\epsilon D_0$ and $v(g'\epsilon M_0) = 1$. $v(g'\epsilon\{z'\epsilon M_0\cup \{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}) = v(A_{M_0\cup\{k\}}(\{k\},\overline{w}_1',\ldots,\overline{w}_\ell')) = v(f' \epsilon\{z'\epsilon M_0\cup\{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\})$. (c) g' cannot be M_0 because $v(k\epsilon M_0) = 0$ and $v(k\epsilon\{k\}) = 1$.

(iii) Let $v(f'\epsilon M_0) = 0$. Then it is not the case that $v(x'\epsilon f') = v(x'\epsilon\{k\})$ for all $x'\epsilon D_0$. (a) Let $g'\epsilon D_1 - (D_0\cup\{M_0\})$. The above holds for g' and hence $v(g'\epsilon M_0) = 0$. Hence $v(f'\epsilon\{z'\epsilon M_0\cup\{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}) = 0 = v(g'\epsilon\{z'\epsilon M_0\cup\{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots, \overline{w}_\ell')\})$. (b) Let g' be M_0. Then $v(g'\epsilon M_0) = 0$ and $v(g'\epsilon\{z'\epsilon M_0\cup\{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}) = 0 = v(f'\epsilon\{z'\epsilon M_0\cup\{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots, \overline{w}_\ell')\})$. (c) Let $g'\epsilon D_0$. Hence g' is $\{k\}$ and $v(x'\epsilon g') = v(x'\epsilon\{k\})$ for all $x'\epsilon D_0$, which yields a contradiction.

This completes the proof.

Let $f'\epsilon D_1 - (D_0\cup\{M_0\})$ and let $v(x'\epsilon f') = v(x'\epsilon\{k\})$ for all $x'\epsilon D_0$. So $v(f'\epsilon M_0) = 1$. Let $g'\epsilon D_1 - D_0$.

(I) Let $v(g'\epsilon M_0) = 1$. Then $v(x'\epsilon g') = v(x'\epsilon\{k\})$, for all $x'\epsilon D_0$. Let the f' above be $\{z'\epsilon M_0\cup\{k\} \ / \ A_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}$. Then

$v(g'\epsilon f') = v(A_{M_0\cup\{k\}}(\{k\}, \overline{w}_1', \ldots, \overline{w}_\ell')) = v(\{k\}\epsilon f') = v(\{k\}\epsilon\{k\}) = 0.$

$v(g'\epsilon\{k\}) = 0 = v(g'\epsilon f').$

(II) Let $v(g'\epsilon M_0) = 0$. Then $v(g'\epsilon f') = 0 = v(g'\epsilon\{k\})$.

Hence, by the Axiom of Extensionality, in all contexts, f' can be replaced by {k}, its corresponding member of D_0. Hence $v(B_{M_0\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell'))$ ($z',\overline{w}_i'$ all ϵD_0) is the same whether the range of the bound variables in $B_{M_0\cup\{k\}}$ is taken as D_0 or D_1.

This completes the initial stage of the transfinite induction. The next step is to assume for some ordinal α that domains D_β, for all $\beta \leqslant \alpha$, have been constructed and that all valuations of the expressions constructed from the members of these domains have been obtained in a way similar to the valuation of expressions from D_1. D_β consists of k, {k}, M_γ, for all γ such that $0 \leqslant \gamma < \beta$, and all expressions of the form: $\{z'\epsilon M_\gamma\cup\{k\} \,/\, A_{M_\gamma\cup\{k\}}(z',\overline{w}_1',\ldots,\overline{w}_\ell')\}$, where $\overline{w}_i'\epsilon D_{\beta-1}$ (if β is a successor ordinal) or $\overline{w}_i'\epsilon D_\beta$ (if β is a limit ordinal) and $v(\overline{w}_i'\epsilon M_\gamma) = 1$ or $\overline{w}_i'$ is k, for all i, for γ such that $0 \leqslant \gamma < \beta$, and where $A_{M_\gamma\cup\{k\}}$ has the value 1 or 0 for all $z'\epsilon D_{\beta-1}$ (if β is a successor ordinal) or $z'\epsilon D_\beta$ (if β is a limit ordinal). The Axiom of Extensionality holds in D_β and if $f'\epsilon D_\beta - D_\gamma$ and $v(f'\epsilon M_\gamma) = 1$ then f' can be replaced by any of its corresponding members, in all contexts with the domain D_β. Also $v(B_{M_\gamma\cup\{k\}}(z',\overline{x}_1',\ldots,\overline{x}_m'))$ is the same whether the range of the bound variables in $B_{M_\gamma\cup\{k\}}$ is taken as D_γ or D_β.

We now define $D_{\alpha+1}$ as all members of the D_β's, for all $\beta \leq \alpha$, M_α, and all expressions of the form: $\{z' \epsilon M_\gamma \cup \{k\} \ / \ A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}$, where $\overline{w}_i' \epsilon D_\alpha$ such that $v(\overline{w}_i' \epsilon M_\gamma) = 1$ or $\overline{w}_i'$ is k, for all i, where $0 \leq \gamma \leq \alpha$, and where $A_{M_\gamma \cup \{k\}}$ has the value 1 or 0 for all $z' \epsilon D_\alpha$.

[If α is a limit ordinal, then $\gamma = \alpha$ is the only case we need to consider].

If g' and $h' \epsilon D_{\alpha+1} - D_\alpha$, then $v(kog') = v(g'ok) = v(g'oh') = n$. If $f' \epsilon D_\alpha - \{k\}$, then $v(g'of') = v(f'og') = n$. Also $v(k \epsilon M_\alpha) = 0$, $v(M_\alpha \epsilon k) = n$ and $v(M_\alpha \epsilon M_\alpha) = 0$. If $f' \epsilon D_\alpha - \{k\}$, then $v(f' \epsilon M_\alpha) = 1$ and $v(M_\alpha \epsilon f') = 0$. If $g' \epsilon D_{\alpha+1} - (D_\alpha \cup \{M_\alpha\})$ then $v(M_\alpha \epsilon g') = 0$, $v(g' \epsilon k) = n$ and $v(g' \epsilon \{k\}) = 0$.

$v(x' \epsilon \{z' \epsilon M_\gamma \cup \{k\} \ / \ A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = v(A_{M_\gamma \cup \{k\}}(x', \overline{w}_1', \ldots, \overline{w}_\ell'))$, for all $x' \epsilon D_\gamma$, where $\overline{w}_i' \epsilon D_\alpha$ such that $v(\overline{w}_i' \epsilon M_\gamma) = 1$ or $\overline{w}_i'$ is k, for all i, and where the range of the bound variables in $A_{M_\gamma \cup \{k\}}$ is taken as D_α for the valuation.

[If α is a limit ordinal, then $\gamma = \alpha$ is the only case we need to consider]. Let $x' \epsilon D_\alpha - D_\gamma$. If $v(x' \epsilon M_\gamma) = 1$, then $v(z' \epsilon x') = v(z' \epsilon y')$ for all $z' \epsilon D_\alpha$, for some $y' \epsilon D_\gamma$, where y' is a corresponding member of D_γ for x'. Then $v(x' \epsilon \{z' \epsilon M_\gamma \cup \{k\} \ / \ A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$ $= v(A_{M_\gamma \cup \{k\}}(y', \overline{w}_1', \ldots, \overline{w}_\ell'))$.

If $v(x' \epsilon M_\gamma) = 0$ then $v(x' \epsilon \{z' \epsilon M_\gamma \cup \{k\} \ / \ A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = 0$.

If $g' \epsilon D_{\alpha+1} - (D_\alpha \cup \{M_\alpha\})$ and $0 \leq \gamma \leq \alpha$, then if $v(x' \epsilon g') = v(x' \epsilon h')$

for all $x'\epsilon D_\alpha$, for some $h'\epsilon D_\gamma$, then $v(g'\epsilon M_\gamma) = 1$ and h' is called a corresponding member of D_γ for g'.

If it is not the case that $v(x'\epsilon g') = v(x'\epsilon h')$ for all $x'\epsilon D_\alpha$, for some $h'\epsilon D_\gamma$, then $v(g'\epsilon M_\gamma) = 0$.

Let $f'\epsilon D_{\alpha+1} - (D_\alpha \cup \{M_\alpha\})$. Let $v(f'\epsilon M_\gamma) = 1$. Then $v(f'\epsilon \{z'\epsilon M_\gamma \cup \{k\} / A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = v(A_{M_\gamma \cup \{k\}}(h', \overline{w}_1', \ldots, \overline{w}_\ell'))$, where h' is a corresponding member of D_γ for f' and where the range of the bound variables is taken as D_α.

Now let $v(f'\epsilon M_\gamma) = 0$. Then $v(f'\epsilon \{z'\epsilon M_\gamma \cup \{k\} / A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = 0$.

Since the Axiom of Extensionality holds in D_α, any corresponding member, h', of D_γ for f' can be substituted in the above expression.

We shall now show that the Axiom of Extensionality holds in $D_{\alpha+1}$.

Let $v(x'\epsilon f') = v(x'\epsilon g')$ for all $x'\epsilon D_{\alpha+1}$, where $f', g'\epsilon D_{\alpha+1} - \{k\}$. $v(f'\epsilon\{k\}) = 0 = v(g'\epsilon\{k\})$. Let $0 \leqslant \gamma \leqslant \alpha$. Let $\overline{w}_i'\epsilon D_\alpha$ and $v(\overline{w}_i'\epsilon M_\gamma) = 1$ or $\overline{w}_i'$ be k, for all i.

(i) Let $v(f'\epsilon M_\gamma) = 1$ and $f'\epsilon D_\gamma$. (a) Let $g'\epsilon D_\gamma$. Then $v(g'\epsilon M_\gamma) = 1$ and $v(g'\epsilon \{z'\epsilon M_\gamma \cup \{k\} / A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = v(A_{M_\gamma \cup \{k\}}(g', \overline{w}_1', \ldots, \overline{w}_\ell')) = v(A_{M_\gamma \cup \{k\}}(f', \overline{w}_1', \ldots, \overline{w}_\ell')) = v(f'\epsilon \{z'\epsilon M_\gamma \cup \{k\} / A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$, using the Axiom of Extensionality in D_α. (b) Let $g'\epsilon D_{\alpha+1} - D_\gamma$. Then f' is a corresponding member of D_γ for g', $v(g'\epsilon M_\gamma) = 1$, and $v(g'\epsilon \{z'\epsilon M_\gamma \cup \{k\} / A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$ $= v(A_{M_\gamma \cup \{k\}}(f', \overline{w}_1', \ldots, \overline{w}_\ell')) = v(f'\epsilon \{z'\epsilon M_\gamma \cup \{k\} / A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$

(ii) Let $v(f'\epsilon M_\gamma) = 1$ and $f'\epsilon D_{\alpha+1} - D_\gamma$. (a) Let $g'\epsilon D_\gamma$. This case has already been treated in (i). (b) Let $g'\epsilon D_{\alpha+1} - D_\gamma$. f' has a corresponding member, j', of D_γ. Hence $v(x'\epsilon f') = v(x'\epsilon j')$ for all $x'\epsilon D_\alpha$, and $v(f'\epsilon\{z'\epsilon M_\gamma \cup \{k\} \,/\, A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = v(A_{M_\gamma \cup \{k\}}(j', \overline{w}_1', \ldots, \overline{w}_\ell'))$. $v(x'\epsilon g') = v(x'\epsilon j')$ for all $x'\epsilon D_\alpha$ and $v(g'\epsilon M_\gamma) = 1$. $v(g'\epsilon\{z'\epsilon M_\gamma \cup \{k\} \,/\, A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = v(A_{M_\gamma \cup \{k\}}(j', \overline{w}_1', \ldots, \overline{w}_\ell')) = v(f'\epsilon\{z'\epsilon M_\gamma \cup \{k\} \,/\, A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$.

(iii) Let $v(f'\epsilon M_\gamma) = 0$. Then it is not the case that $v(x'\epsilon f') = v(x'\epsilon j')$ for all $x'\epsilon D_\alpha$, for some $j'\epsilon D_\gamma$. (a) Let $g'\epsilon D_\gamma$. Then $v(x'\epsilon f') = v(x'\epsilon g')$ for all $x'\epsilon D_\alpha$. This is a contradiction. (b) Let $g'\epsilon D_{\alpha+1} - D_\gamma$. Then it is not the case that $v(x'\epsilon g') = v(x'\epsilon j')$ for all $x'\epsilon D_\alpha$, for some $j'\epsilon D_\gamma$. Hence $v(g'\epsilon M_\gamma) = 0$ and $v(g'\epsilon\{z'\epsilon M_\gamma \cup \{k\} \,/\, A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}) = 0 = v(f'\epsilon\{z'\epsilon M_\gamma \cup \{k\} \,/\, A_{M_\gamma \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\})$.

This completes the proof.

Let $f'\epsilon D_{\alpha+1} - D_\gamma$ and let $v(x'\epsilon f') = v(x'\epsilon h')$ for all $x'\epsilon D_\alpha$, for some $h'\epsilon D_\gamma$ $(0 \leqslant \gamma \leqslant \alpha)$. So $v(f'\epsilon M_\gamma) = 1$. This covers all cases of $v(f'\epsilon M_\gamma) = 1$, where $f'\epsilon D_\alpha - D_\gamma$, because f' and h' can be interchanged in all contexts in D_α. Also f' cannot take the form M_δ, $\gamma \leqslant \delta \leqslant \alpha$, because $v(M_\delta \epsilon M_\gamma) = 0$. Let f' be $\{z'\epsilon M_\kappa \cup \{k\} \,/\, A_{M_\kappa \cup \{k\}}(z', \overline{w}_1', \ldots, \overline{w}_\ell')\}$, where $0 \leqslant \kappa \leqslant \alpha$ and $\overline{w}_i'\epsilon D_\alpha$ and $v(\overline{w}_i'\epsilon M_\kappa) = 1$ or $\overline{w}_i$ is k, for all i.

Let $g'\epsilon D_{\alpha+1} - D_\alpha$.

(I) Let $v(g' \epsilon M_{\kappa}) = 1$. Then $v(x' \epsilon g') = v(x' \epsilon i')$ for all $x' \epsilon D_{\alpha}$, for some $i' \epsilon D_{\kappa}$. Then $v(g' \epsilon f') = v(A_{M_{\kappa} \cup \{k\}}(i', \overline{w}_1', \ldots, \overline{w}_{\ell}')) = v(i' \epsilon f') = v(i' \epsilon h')$.

Let h' be $\{z' \epsilon M_{\delta} \cup \{k\} \ / \ B_{M_{\delta} \cup \{k\}}(z', \overline{x}_1', \ldots, \overline{x}_m')\}$, where $0 \leqslant \delta < \gamma \leqslant \alpha$ and $\overline{x}_i' \epsilon D_{\gamma-1}$ (or D_{γ}, if γ is a limit ordinal) such that $v(\overline{x}_i' \epsilon M_{\delta}) = 1$ or $\overline{x}_i'$ is k, for all i.

(i) Let $v(g' \epsilon M_{\delta}) = 1$. Then $v(x' \epsilon g') = v(x' \epsilon j')$ for all $x' \epsilon D_{\alpha}$, for some $j' \epsilon D_{\delta}$. Hence $v(g' \epsilon h') = v(B_{M_{\delta} \cup \{k\}}(j', \overline{x}_1', \ldots, \overline{x}_m')) = v(j' \epsilon h')$.

(a) Let $\delta \leqslant \kappa$. Then $D_{\delta} \subseteq D_{\kappa}$ and j' could have been used as the corresponding member of D_{κ} for g' and hence $v(g' \epsilon f') = v(j' \epsilon h') = v(g' \epsilon h')$.

(b) Let $\delta > \kappa$. Then $D_{\kappa} \subseteq D_{\delta}$ and i' could have been used as the corresponding member of D_{δ} for g' and hence $v(g' \epsilon h') = v(i' \epsilon h') = v(g' \epsilon f')$.

(ii) Let $v(g' \epsilon M_{\delta}) = 0$. Then $v(g' \epsilon h') = 0$. It is not the case that $v(x' \epsilon g') = v(x' \epsilon j')$ for all $x' \epsilon D_{\alpha}$, for some $j' \epsilon D_{\delta}$. This also follows for i'. Hence $v(i' \epsilon M_{\delta}) = 0$ and $v(i' \epsilon h') = 0$. Hence $v(g' \epsilon f') = 0 = v(g' \epsilon h')$.

Let h' be $\{k\}$. Then $v(g' \epsilon h') = 0$ and $v(g' \epsilon f') = v(i' \epsilon h') = 0$.

Let h' be M_{δ}, $0 \leqslant \delta < \gamma \leqslant \alpha$. As above, if $v(g' \epsilon M_{\delta}) = 1$ (or 0) then $v(i' \epsilon M_{\delta}) = 1$ (or 0) and hence $v(g' \epsilon f') = v(g' \epsilon h')$.

(II) Let $v(g' \epsilon M_{\kappa}) = 0$. Then it is not the case that $v(x' \epsilon g') = v(x' \epsilon i')$ for all $x' \epsilon D_{\alpha}$, for some $i' \epsilon D_{\kappa}$. $v(g' \epsilon f') = 0$.

Let h' be $\{z' \in M_{\xi} \cup \{k\} \;/\; B_{M_{\xi} \cup \{k\}}(z', \bar{x}_1', \ldots, \bar{x}_m')\}$, where $0 \leq \xi < \gamma \leq \alpha$ and $\bar{x}_i' \in D_{\gamma-1}$ (or D_{γ}, if γ is a limit ordinal) such that v($\bar{x}_i' \in M_{\xi}$)=1 or $\bar{x}_i'$ is k, for all i.

(i) Let v($g' \in M_{\delta}$)=1. Then v($x' \in g'$)=v($x' \in j'$) for all $x' \in D_{\alpha}$, for some $j' \in D_{\delta}$. If $\delta \leq \kappa$, then $D_{\delta} \subseteq D_{\kappa}$ and v($x' \in g'$)=v($x' \in j'$) for all $x' \in D_{\alpha}$, for some $j' \in D_{\kappa}$, which yields a contradiction. Hence, let $\delta > \kappa$. v($g' \in h'$)=v($j' \in h'$)=v($j' \in f'$). It is not the case that v($x' \in j'$) =v($x' \in i'$) for all $x' \in D_{\alpha}$, for some $i' \in D_{\kappa}$. Hence v($j' \in M_{\kappa}$)=0 and v($j' \in f'$)=0. Hence v($g' \in h'$)=0=v($g' \in f'$).

(ii) Let v($g' \in M_{\delta}$)=0. Then v($g' \in h'$)=0=v($g' \in f'$).

Let h' be $\{k\}$. Then v($g' \in h'$)=0=v($g' \in f'$).

Let h' be M_{δ}, $0 \leq \delta < \gamma \leq \alpha$. Let v($g' \in M_{\delta}$)=1. Then v($x' \in g'$)=v($x' \in j'$) for all $x' \in D_{\alpha}$, for some $j' \in D_{\delta}$. If $\delta \leq \kappa$, then $D_{\delta} \subseteq D_{\kappa}$ and v($x' \in g'$) =v($x' \in j'$) for all $x' \in D_{\alpha}$, for some $j' \in D_{\kappa}$, which yields a contradiction. Hence, let $\delta > \kappa$. v($j' \in M_{\delta}$)=1=v($j' \in f'$). It is not the case that v($x' \in j'$)=v($x' \in i'$) for all $x' \in D_{\alpha}$, for some $i' \in D_{\kappa}$. Hence v($j' \in M_{\kappa}$)=0. Hence v($j' \in f'$)=0, which is a contradiction. Hence v($g' \in M_{\delta}$) =0 and v($g' \in h'$)=0=v($g' \in f'$).

Hence, by the Axiom of Extensionality, in all contexts, f' can be replaced by a corresponding member of D_{γ}, $0 \leq \gamma \leq \alpha$. Hence v($B_{M_{\gamma} \cup \{k\}}$ (z', $\bar{x}_1', \ldots, \bar{x}_m'$)) ($z'$, $\bar{w}_i'$ all $\in D_{\alpha}$) is the same whether the range of the bound variables in $B_{M_{\gamma} \cup \{k\}}$ is D_{γ} or $D_{\alpha+1}$, where $0 \leq \gamma \leq \alpha$.

The next stage of the transfinite induction is to consider the formation of D_{α}, α a limit ordinal. D_{α} consists of k, $\{k\}$, M_{γ}, for all γ such that $0 \leq \gamma < \alpha$, and all expressions of the form : $\{z' \in M_\gamma \cup \{k\} / A_{M_\gamma \cup \{k\}}(z', \bar{w}_1', \ldots, \bar{w}_\ell')\}$, where $\bar{w}_i' \in D_\alpha$ and $v(\bar{w}_i' \in M_\gamma) = 1$ or $\bar{w}_i'$ is k, for all i, where $0 \leq \gamma < \alpha$, and where $A_{M_\gamma \cup \{k\}}$ has the value 1 or 0 for all $z' \in D_\alpha$. That is, $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$. By the induction hypothesis, all the valuations for D_α have been made, the Axiom of Extensionality holds in D_α and if $f' \in D_\alpha - D_\gamma$ and $v(f' \in M_\gamma) = 1$ then f' can be replaced by any of its corresponding members, in all contexts with the domain D_α. Also $v(B_{M_\gamma \cup \{k\}}(z', \bar{x}_1', \ldots, \bar{x}_m'))$ is the same whether the range of the bound variables in $B_{M_\gamma \cup \{k\}}$ is taken as D_γ or D_α.

Now define $D^U = \bigcup_\alpha D_\alpha$. D^U consists of k, $\{k\}$, M_α, for all α, and all expressions of the form : $\{z' \in M_\alpha \cup \{k\} / A_{M_\alpha \cup \{k\}}(z', \bar{w}_1', \ldots, \bar{w}_\ell')\}$, where $\bar{w}_i' \in D^U$ and $v(\bar{w}_i' \in M_\alpha) = 1$ or $\bar{w}_i'$ is k, for all i, where α is any ordinal, and where $A_{M_\alpha \cup \{k\}}$ has the value 1 or 0 for all $z' \in D^U$. By the transfinite induction, all the valuations for D^U have been made, the Axiom of Extensionality holds in D^U, if $f' \in D^U - D_\alpha$ and $v(f' \in D_\alpha) = 1$ then f' can be replaced by any of its corresponding members, in all contexts with the domain D^U, and $v(A_{M_\alpha \cup \{k\}}(z', \bar{w}_1', \ldots, \bar{w}_\ell'))$ is the same whether the range of the bound variables in $A_{M_\alpha \cup \{k\}}$ is taken as D_α or D^U.

The following valuations hold in D^U :

If $f' \in D^U - \{k\}$ and $g' \in D^U - \{k\}$ then $v(kok) = 1$, $v(f'ok) = v(kof') = v(f'og') = n$, $v(k \in M_\alpha) = 0$, $v(M_\beta \in M_\alpha) = 1$ if $\beta < \alpha$, $v(M_\gamma \in M_\alpha) = 0$ if $\gamma \geq \alpha$. If $x' \in D^U$ then $v(x' \in k) = n$. If $g' \in D^U - k$, then $v(g' \in \{k\}) = 0$ and $v(k \in \{k\}) = 1$.

If f' is $\{z' \epsilon M_\alpha \cup \{k\} \ / \ A_{M_\alpha \cup \{k\}}(z', \bar{w}_1', \ldots, \bar{w}_k')\}$, where $\bar{w}_i' \epsilon D^U$ and $v(\bar{w}_i \epsilon M_\alpha)=1$ or $\bar{w}_i'$ is k, for all i, then $v(f' \epsilon M_\delta)=1$ for all $\delta > \alpha$, and $v(M_\tau \epsilon f')=0$ for all $\tau \geq \alpha$. If $v(x' \epsilon M_\alpha)=1$ or x' is k, then $v(x' \epsilon f')=v(A_{M_\alpha \cup \{k\}}(x', \bar{w}_1', \ldots, \bar{w}_k'))$. If $v(x' \epsilon M_\alpha)=0$ and x' is not k, then $v(x' \epsilon f')=0$. $v(\{k\} \epsilon M_\alpha)=1$, for all α. If $v(x' \epsilon f')=v(x' \epsilon h')$ for all $x' \epsilon D^U$, for some h' such that $v(h' \epsilon M_\tau)=1$ for some $0 \leq \tau \leq \alpha$, then $v(f' \epsilon M_\tau)=1$. If it is not the case that $v(x' \epsilon f')=v(x' \epsilon h')$ for all $x' \epsilon D^U$, for some h' such that $v(h' \epsilon M_\tau)=1$, for some $0 \leq \tau \leq \alpha$, then $v(f' \epsilon M_\tau)=0$. If $v(x' \epsilon M_\delta)=1$ then $v(x' \epsilon M_\kappa)=1$, for all $x' \epsilon D^U$, for all δ and κ such that $\delta \leq \kappa$. Hence for any $x' \epsilon D^U$, except for k, there is a least ordinal α such that $v(x' \epsilon M_\alpha)=1$. Hence $v(x' \epsilon M_\gamma)=0$, for all $\gamma < \alpha$ and $v(x' \epsilon M_\gamma)=1$ for all $\gamma \geq \alpha$. Call this least ordinal, $\underline{\alpha_{x'}}$. Note that $\alpha_{M_\beta}=\beta+1$, $\alpha_{\{k\}}=0$, and $\alpha_{x'}$ is always a successor ordinal. If $\alpha_{f'} \leq \alpha_{g'}$, then $v(g' \epsilon f')=0$. If f' is $\{z' \epsilon M_\alpha \cup \{k\} \ / \ A_{M_\alpha \cup \{k\}}(z', \bar{w}_1', \ldots, \bar{w}_k')\}$, then $\alpha_{f'} \leq \alpha+1$. If $v(g' \epsilon f')=1$ then $\alpha_{g'} \leq \alpha_{f'}-1$.

The domain D^U and its valuations will form the model for the axioms involving sets and individuals, which will be shown later. We now construct a domain D^S which, with its valuations, will form the model for the axioms involving special classes and individuals.

Let D^O consist of all the members of D^U and all the expressions of the form : $\{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m')\}$, where $\bar{x}_i' \epsilon D^U$, for all i, where A contains quantification only over sets and individuals, and where A has the value 1 or 0 for all $z' \epsilon D^U$. In the following, all quantification over sets and individuals that occurs in predicates A will be evaluated over D^U.

If $\{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m')\} \in D^0 - D^U$, then $v(y' \in \{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m')\}) = v(A(y', \bar{x}_1', \ldots, \bar{x}_m'))$, for all $y' \in D^U$. If $v(y' \in \{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m')\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m')\} \in u) = v(w' \in u)$, for all $u \in D^0$. If it is not the case that $v(y' \in \{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m')\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m')\} \in u) = 0$, for all $u \in D^0$.

Given that D^n and its valuations have been determined, D^{n+1} consists of all of the members of D^n and all the expressions of the form : $\{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}$, where $\bar{x}_i' \in D^U$, for all i, and $\bar{y}_j \in D^n$, for all j, where A contains quantification only over sets and individuals, and where A has the value 1 or 0 for all $z' \in D^U$.

If $\{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\} \in D^{n+1} - D^n$ then $v(y' \in \{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}) = v(A(y', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p))$, for all $y' \in D^U$. If $v(y' \in \{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\} \in u) = v(w' \in u)$, for all $u \in D^{n+1}$. If it is not the case that $v(y' \in \{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z' \,/\, A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\} \in u) = 0$ for all $u \in D^{n+1}$. If $v(y' \in v) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, where $v \in D^n - D^U$, then $v(v \in u) = v(w' \in u)$ for all

$u \in D^{n+1} - D^n$. If it is not the case that $v(y' \in v) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(v \in u) = 0$, for all $u \in D^{n+1} - D^n$.

We need to show that if $v(y' \in w') = v(y' \in w_1')$ for all $y' \in D^U$, then $v(w' \in u) = v(w_1' \in u)$, for all $u \in D^{n+1} - D^n$, where w' and $w_1' \in D^U$. By the Axiom of Extensionality for D^U, the above holds for all $u \in D^0$. Let us assume that the above holds for $u \in D^n$. Now let $u \in D^{n+1} - D^n$. For some predicate A, u is $\{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}$. $v(w' \in u) = v(A(w', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p))$. If $\bar{y}_i \in D^n - D^U$, then either $v(\bar{y}_i \in w') = v(\bar{y}_i \in w_1') = 0$ or $v(\bar{y}_i \in w') = v(\bar{y}_i' \in w') = v(\bar{y}_i' \in w_1') = v(\bar{y}_i \in w_1')$, for some $\bar{y}_i' \in D^U$. Hence $v(A(w', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)) = v(A(w_1', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p))$ and $v(w' \in u) = v(w_1' \in u)$.
This completes the proof.

Let $D^S = \bigcup_n D^n$ and let D^S have the valuations obtained by induction on the D^n's. Hence D^S consists of all the members of D^U and all expressions of the form : $\{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}$, where $\bar{x}_i' \in D^U$, for all i, and $\bar{y}_j \in D^S$, for all j, where A contains quantification over sets and individuals only, and where A has the value 1 or 0 for all $z' \in D^U$.

If $\{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\} \in D^S - D^U$, then $v(y' \in \{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}) = v(A(y', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p))$, for all $y' \in D^U$. If $v(y' \in \{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\} \in u) = v(w' \in u)$, for all $u \in D^S$. If it is not the case that $v(y' \in \{z' \ / \ A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\}) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(\{z' / A(z', \bar{x}_1', \ldots, \bar{x}_m', \bar{y}_1, \ldots, \bar{y}_p)\} \in u)$ $=0$, for all $u \in D^S$. If $v(y' \in w') = v(y' \in w_1')$ for all $y' \in D^U$, then $v(w' \in u) = v(w_1' \in u)$ for all $u \in D^S$, where w' and $w_1' \in D^U$. This follows by induction using an above argument.

Now we will show that D^S and its valuations form a model, N', for all the axioms.

The domain for special classes and individuals is D^S, the domain for individuals is $\{k\}$, the domain for sets and individuals is D^U, the domain for special classes is $D^S - \{k\}$, and the domain for sets is $D^U - \{k\}$.

The General Axioms 1, 2 and 3 are obviously valid in the model N'. Individual Axiom 1 is valid because there is only one individual, k, in the model. Individual Axiom 2 is valid because the fusion of f' is k. Individual Axiom 3 is valid because f' is either $\{k\}$ or 0, where 0 can be taken as $\{z' \in M_0 \cup \{k\} / \sim(k \in \{k\})\}$. Individual Axioms 4 and 5 are valid because there is one individual, k.

For showing the validity of Axiom T, let $v(x \in f) = v(x \in g)$ for all $x \in D^S$, where f and $g \in D^S$. If f and $g \in D^U$, then we have already shown that $v(f \in u) = v(g \in u)$, for all $u \in D^S$. If $f \in D^S - D^U$ and $g \in D^U$, then by the construction of D^S, $v(f \in u) = v(g \in u)$ for all $u \in D^S$. Similarly, if $f \in D^U$ and $g \in D^S - D^U$. Let $f \in D^S - D^U$ and $g \in D^S - D^U$. (a) If $v(y' \in f) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$, then $v(y' \in g) = v(y' \in w')$ for all $y' \in D^U$, for some $w' \in D^U$. Hence $v(f \in u) = v(w' \in u) = v(g \in u)$, for all $u \in D^S$.

(b) If it is not the case that $v(y'\in f)=v(y'\in w')$ for all $y'\in D^U$, for some $w'\in D^U$, then it is not the case that $v(y'\in g)=v(y'\in w')$ for all $y'\in D^U$, for some $w'\in D^U$. Hence $v(f\in u)=0=v(g\in u)$, for all $u\in D^S$. Hence Axiom T is valid in the model.

To show that the Axiom P is valid in the model, let x' and y' be unequal to k. Let $\alpha_{x'}\leq\alpha_{y'}$. Then $v(x'\in M_{\alpha_{y'}})=v(y'\in M_{\alpha_{y'}})=1$. The required f' is then $\{z'\in M_{\alpha_{y'}}\cup\{k\}$ / $(Aw')(w'\in M_{\alpha_{y'}}\supset T(w'\in z'\doteq w'\in x'))$ & $T(k\in z'\doteq k\in x')$.v.$(Aw')(w'\in M_{\alpha_{y'}}\supset T(w'\in z'\doteq w\in y'))$ & $T(k\in z'\doteq k\in y')\}$. Now let x' be unequal to k and let y' be k. Then the required f' is $\{z'\in M_{\alpha_{x'}}\cup\{k\}$ / $z'\in\{k\}$ v.$(Aw')(w'\in M_{\alpha_{x'}}\supset T(w'\in z'\doteq w'\in x'))$ & $T(k\in z'\doteq k\in x')\}$. If x' and y' are both k, then the required f' is $\{k\}$.

Axiom N is valid as the required f' can be taken as $\{z'\in M_0\cup\{k\}$ / $\sim(k\in\{k\})\}$. As before, call this O.

For showing the validity of Axiom B, consider the predicate $\phi(x_1',\ldots,x_\ell',y_1,\ldots,y_m)$, where only variables over sets and individuals are quantified, and where ϕ is significant for all substitutions into its free variables. The required f' is $\{z'$ / $(Sx_1')\ldots(Sx_\ell')((Aw')T(w'\in z'\doteq w'\in\langle x_1',\ldots,x_\ell'\rangle)$ & $\phi(x_1',\ldots,x_\ell',\bar{y}_1,\ldots,\bar{y}_m))\}$. The $\langle x_1',\ldots,x_\ell'\rangle$ is defined the same way as earlier in this chapter, using the set f', used to show the validity of Axiom P.

For the Axiom U, the required f' is $\{z'\in M_{\alpha_{f'}}\cup\{k\}$ / $(Sv')(T_n(v'$

$\in M_{\alpha_{f'}}$) & $z'\in v'$ & $v'\in f'$) v.$z'\in k$ & $k\in f'$}.

For the Axiom W, consider all the members x' of f' in the model, i.e. such that $v(x'\in f')=1$. All of these members will be members of $M_{\alpha_{f'}-1}$, i.e. such that $v(x'\in M_{\alpha_{f'}-1})=1$, or be k. Consider a set S of only those members of f' which are members of the set $D_{\alpha_{f'}-1}$. Any member of f' not in $D_{\alpha_{f'}-1}$ will be replaceable by a member of f' in $D_{\alpha_{f'}-1}$ in all contexts. Let the set T be the set of all subsets of S. Form a subset R of T such that $X\in R \equiv X\in T$ & $(Sh')(h'\in D^U$ & $(Aw')(w'\in D^U \supset. v(w'\in h')=1 \equiv w'\in X))$. [By transfinite induction, the class V of ordered pairs $\langle x',y'\rangle$, $x',y'\in D^U$, such that $v(x'\in y')=1$ can be constructed so that '$v(w'\in h')=1$' can be replaced by '$\langle w',h'\rangle\in V$'.] For each member X of R, since there is a member h' of D^U there is a least ordinal $\alpha_{h'}$ such that $v(h'\in M_{\alpha})$ $=1$. Hence choose a h' from $D_{\alpha_{h'}}$ satisfying the above property. So to each member X of R we can choose a corresponding h' from D^U. Since there is a set of such h' 's, there is an ordinal β which is the sup of all ordinals $\alpha_{h'}$. Hence the required g' can be taken as $\{z'\in M_{\beta}\cup\{k\}$ / $(Aw')(w'\in M_{\beta} \supset. T(w'\in z' \supset w'\in f'))$ & $T(k\in z'$ $\supset k\in f')\}$. This is the required power set of f' because, by the above argument, all possible subsets of f' will be members of M_{β}.

Next we will show that Axiom R is valid but firstly in a form applicable to sets and individuals only. That is, if $A(x', y', u_1',\ldots,u_m')$ is univocal then $(Sg')(Ay')(y'\in g' \equiv (Sx')(A(x', y',$

$u_1',\ldots,u_m'$) & $x'\in f'$)), where quantification in A is over sets and individuals only, and where A is significant for all substitutions into its free variables. By Theorem 1, we need only consider wffs A such that A contains only the connectives ~, & and T and the quantifier A. By Meta-theorem 7 of Chapter 2, we need only consider wffs A such that A has all of its quantifiers, A and E, at the beginning of the formula. [In the proof of Meta-theorem 7, Sp can be defined as ~(~Tp & ~T~p) and $T(Ex)A(x) \simeq (Ex)(Ay)$ $(TA(x)$ & $SA(y))$.] The proof will follow that in [3], pp.90-92.

Lemma 1.

Let $y'=\phi(x')$ be a univocal function defined by a formula $A(x',$ $y', u_1',\ldots,u_m')$ for some $u_i'\in D^U$ and such that $x'\in D^U$ implies $\phi(x')$ $\in D^U$. If $u'\in D^U$ then there is a $w'\in D^U$ such that if v' is the range of ϕ on u', then $y'\in v' \supset. v(y'\in w')=1$, for all $y'\in D^U$. [v' is a set of members of D^U, and cannot belong to D^U.]

Proof. Note that if $v(w_1'\in z')=v(w_1'\in x')$ for all $w_1'\in D^U$ and where z' and $x'\in D^U$, then $v(w_2'\in\phi(z'))=v(w_2'\in\phi(x'))$, for all $w_2'\in D^U$. For each x' such that $v(x'\in u')=1$ and $x'\in D_{\alpha_{u'}-1}$, let $g(x')$ be the least ordinal α such that $v(\phi(x')\in M_\alpha)=1$, if $\phi(x')\in D^U-\{k\}$, and let $g(x')$ be 0 if $\phi(x')$ is k. Let β be the sup of all these $g(x')$'s. Clearly $y'\in v' \supset v(y'\in M_\beta\cup\{k\})=1$, for all $y'\in D^U$. [$M_\beta\cup\{k\}$ can be taken as $\{z'\in M_\beta\cup\{k\}$ / $k\in\{k\}\}$.]

Lemma 2.

Let $A(x_1',\ldots,x_n')$ be a wff with the above restrictions on connectives and quantifiers and with its quantifiers, A and E, at the beginning of the formula. Let $\bar{y}' \in D^U - \{k\}$. There is an $M_\mu \cup \{k\} \in D^U$ such that if $v(z' \in \bar{y}')=1$ then $v(z' \in M_\mu \cup \{k\})=1$, for all $z' \in D^U$, and for all $\bar{x}_i' \in D^U$ such that $v(\bar{x}_i' \in M_\mu \cup \{k\})=1$, $v(A(\bar{x}_1', \ldots, \bar{x}_n'))=v(A_{M_\mu \cup \{k\}}(\bar{x}_1', \ldots, \bar{x}_n'))$, where $A_{M_\mu \cup \{k\}}$ is A with all its bound variables restricted to $M_\mu \cup \{k\}$. [$M_\mu \cup \{k\}$ can be taken as $\{z' \in M_\mu \cup \{k\} / k \in \{k\}\}$.]

<u>Proof.</u> By the above conditions, A is of the form : $Q_1 y_1' \ldots Q_m y_m'$ $B(x_1', \ldots, x_n', y_1', \ldots, y_m')$, where Q_r $(1 \leq r \leq m)$ is either A or E. Let $\bar{u}' \in D^U - k$. For $1 \leq r \leq m$ there are functions $f_r(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r-1})$ defined for $\bar{x}_i'$, $\bar{y}_j'$ in $\bar{u}'$ (i.e. $v(\bar{x}_i' \in \bar{u}')=1$ and $v(\bar{y}_j' \in \bar{u}')=1$.) with the following property : If Q_r is E and there is a set or individual $\bar{y}_r' \in D^U$ such that :

(1) ______ $Q_{r+1} y_{r+1}' \ldots Q_m y_m'$ $B(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r', \bar{y}_{r+1}', \ldots, y_m')$ is non-significant in the model, or given that there are no sets or individuals $\bar{y}_r'$ in D^U such that (1) is non-significant in the model, there is a set or individual $\bar{y}_r'$ in D^U such that (1) is true in the model, then $f_r = \alpha$ where α is the least ordinal such that there is a $\bar{y}_r' \in M_\alpha \cup \{k\}$ (i.e. $v(\bar{y}_r' \in M_\alpha \cup \{k\})=1$) satisfying either of the above conditions. If no such $\bar{y}_r'$ exists, put $f_r = 0$. If Q_r is A (i.e. ~E~), then f_r is defined the same way as for E except that (1) is replaced by its negation.

Let β be the sup of $f_r(\bar{x}_i', \bar{y}_j')$ for all $\bar{x}_i'$ and $\bar{y}_j' \in D_{\alpha_{\bar{u}'}-1}$, such that $v(\bar{x}_i' \in \bar{u}')=1$ and $v(\bar{y}_j' \in \bar{u}')=1$, and all r for $1 \leq r \leq m$.

Put $(\bar{u}')^* = \bar{u}' \cup M_\beta \cup \{k\}$. This union can be formed using the Pairing Axiom and Sum Set Axiom which have already been shown to be valid in the model. So $(\bar{u}')^* \in D^U$. Now we define a sequence $\bar{z}_n'$ with $\bar{z}_0'$ as $M_\alpha \cup \{k\}$, if α is the least ordinal such that $v(\bar{y}' \in M_\alpha)=1$, and $\bar{z}_{n+1}' = (\bar{z}_n')^*$. So $\bar{z}_n' \in D^U$, for all n. Let $\bar{z}' = \bigcup_n \bar{z}_n'$. This requires the validity of the Axiom of Infinity, which will be shown later. Assuming this, $\bar{z}' \in D^U$. So $\bar{z}' = \bigcup\{M_\beta / \beta < \alpha'\} \cup \{k\}$, for some α'. $\bar{z}' = M_{\alpha'} \cup \{k\}$ if α' is a limit ordinal or $\bar{z}' = M_{\alpha'-1} \cup \{k\}$ if α' is a successor ordinal. If $v(x' \in M_\alpha)=1$ then $v(x' \in \bar{z}')=1$ for all $x' \in D^U$, and hence if $v(x' \in \bar{y}')=1$ then $v(x' \in \bar{z}')=1$ for all $x' \in D^U$.

Now we need to show that $v(A(\bar{x}_1',\dots,\bar{x}_n'))=v(A_{\bar{z}'}(\bar{x}_1',\dots,\bar{x}_n'))$ for all $\bar{x}_i' \in D^U$ such that $v(\bar{x}_i' \in \bar{z}')=1$. Let $C(\bar{x}_1',\dots,\bar{x}_n', \bar{y}_1',\dots, \bar{y}_r')$ denote the statement (1). Assume that we have shown that, for $r > r_0$, $C \simeq C_{\bar{z}'}$ is true in the model, for all $\bar{x}_i'$ and $\bar{y}_j' \in D^U$ such that $v(\bar{x}_i' \in \bar{z}')=1$ and $v(\bar{y}_j' \in \bar{z}')=1$. Certainly this is the case for $r_0=m$. Then with $r=r_0$, given that $v(\bar{x}_i' \in \bar{z}')=1$ and $v(\bar{y}_j' \in \bar{z}')=1$, they must all lie in $\bar{z}_k'$ for some k.

Let Q_{r+1} be E. Then if $C(\bar{x}_1',\dots,\bar{x}_n', \bar{y}_1',\dots,\bar{y}_r')$ is true in the model then there is a $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}_{k+1}')=1$ and $C(\bar{x}_1',\dots,\bar{x}_n', \bar{y}_1',\dots,\bar{y}_{r+1}')$ is true in the model and for all $\bar{y}_{r+1}' \in D^U$, $C(\bar{x}_1',\dots,\bar{x}_n', \bar{y}_1',\dots,\bar{y}_{r+1}')$ is significant in the model. By assumption, $C_{\bar{z}'}(\bar{x}_1',\dots,\bar{x}_n', \bar{y}_1',\dots,\bar{y}_{r+1}')$ is true for the chosen $\bar{y}_{r+1}'$ and significant for all $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}')=1$, and hence $C_{\bar{z}'}(\bar{x}_1',\dots,\bar{x}_n', \bar{y}_1',\dots,\bar{y}_r')$ is true.

If $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is false (in the model) then for all $\bar{y}_{r+1}' \in D^U$, $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is false. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1, \ldots, \bar{y}_{r+1}')$ is false for all $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}')=1$ and hence $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is false. If $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is non-significant (in the model) then there is a $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}_{k+1}')=1$ and $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is non-significant. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is non-significant for the chosen $\bar{y}_{r+1}'$, and hence $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is non-significant.

Let Q_{r+1} be A (i.e. $\sim$E$\sim$). If $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is true (in the model) then for all $\bar{y}_{r+1}' \in D^U$, $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is true. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is true for all $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}')=1$ and hence $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is true. If $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is false (in the model) then there is a $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}_{k+1}')=1$ and $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is false and , for all $\bar{y}_{r+1}' \in D^U$, $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is significant. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is false for the chosen $\bar{y}_{r+1}'$ and significant for all $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}')=1$, and hence $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is false. If $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is non-significant (in the model) then there is a $\bar{y}_{r+1}' \in D^U$ such that $v(\bar{y}_{r+1}' \in \bar{z}_{k+1}')=1$ and $C(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_{r+1}')$ is non-significant. By assumption, $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n',$

$\bar{y}_1', \ldots, \bar{y}_{n+1}'$) is non-significant for the chosen $\tilde{\bar{y}}_{r+1}'$, and hence $C_{\bar{z}'}(\bar{x}_1', \ldots, \bar{x}_n', \bar{y}_1', \ldots, \bar{y}_r')$ is non-significant. This completes the proof.

Theorem 2.

The Axiom of Replacement in the form : $(Ax')(S!y')A(x', y', u_1', \ldots, u_m')$ & $(Ax', y', u_1', \ldots, u_m')SA(x', y', u_1', \ldots, u_m') \supset (Sg')(Ay')(y' \in g' \equiv (Sx')(A(x', y', u_1', \ldots, u_m')$ & $x' \in f'))$, where A contains quantification over sets and individuals only, is valid in the model.

Proof. Let $A(x', y', u_1', \ldots, u_m')$ define a univocal function in D^U : $y' = \phi(x')$, for particular $\bar{u}_1', \ldots, \bar{u}_m'$ in D^U. Let $\bar{f}' \in D^U$ and let $\bar{v}'$ be the range of ϕ on $\bar{f}'$. [$\bar{v}'$ is a set of members of D^U but does not itself belong to D^U.] By lemma 1, there is an α such that $z' \in \bar{v}' \supset . v(z' \in M_\alpha \cup \{k\}) = 1$, for all $z' \in D^U$. We can assume that $\bar{f}', \bar{u}_1', \ldots, \bar{u}_m'$ all belong to $M_\alpha \cup \{k\}$, i.e. $v(\bar{f}' \in M_\alpha \cup \{k\}) = 1$ and $v(\bar{u}_i' \in M_\alpha \cup \{k\}) = 1$ for all i. Taking $M_\alpha \cup \{k\}$ as the $\bar{y}'$ of lemma 2, it follows that for some μ, $v(A(x', y', \bar{u}_1', \ldots, \bar{u}_m')) = v(A_{M_\mu \cup \{k\}}(x', y', \bar{u}_1', \ldots, \bar{u}_m'))$, for all $x', y' \in D^U$ such that $v(x' \in M_\mu \cup \{k\}) = 1$ and $v(y' \in M_\mu \cup \{k\}) = 1$. Also $v(z' \in M_\alpha \cup \{k\}) = 1$ implies that $v(z' \in M_\mu \cup \{k\}) = 1$, for all $z' \in D^U$. Hence, for all $z' \in D^U$, $z' \in \bar{v}' \supset v(z' \in M_\mu \cup \{k\}) = 1$. Also, $v(\bar{f}' \in M_\mu \cup \{k\}) = 1$ and $v(\bar{u}_i' \in M_\mu \cup \{k\}) = 1$, for all i. Hence the required $\bar{g}'$ can be taken as $\{y' \in M_\mu \cup \{k\} / (Sx')(T_n(x' \in M_\mu)$ & $x' \in \bar{f}'$ & $A_{M_\mu \cup \{k\}}(x', y', \bar{u}_1', \ldots, \bar{u}_m'))$ v $(k \in \bar{f}'$ & $A_{M_\mu \cup \{k\}}(k, y', \bar{u}_1', \ldots, \bar{u}_m'))\}$. For arbitrary $\bar{u}_1', \ldots, \bar{u}_m'$, $\bar{f}'$, an ordinal μ can be found so that the above $\bar{g}'$ represents the set $\bar{v}'$ in D^U, in that $z'\in\bar{v}'$ iff $v(z'\in\bar{g}')=1$, for all $z'\in D^U$. Hence the above form of the Axiom of Replacement is valid in the model.

Lemma 3.

If $f\in D^S-D^U$, then there is a $g\in D^0$ such that , for all $z\in D^S$, $v(z\in f)$ $=v(z\in g)$.

Proof. Let $f\in D^{n+1}-D^n$ and assume that the lemma holds for all members of D^n. Let f be $\{z' \ / \ A(z', \bar{u}_1',\ldots,\bar{u}_m', \bar{v}_1,\ldots,\bar{v}_\ell)\}$, where $\bar{u}_i'\in D^U$, for all i, and $\bar{v}_j\in D^n$, for all j. By the assumption, for each $\bar{v}_j\in D^n$ there is a $\bar{w}_j\in D^0$ such that $v(z\in\bar{v}_j)=v(z\in\bar{w}_j)$ for all $z\in D^S$. Let f_1 be $\{z' \ / \ A(z', \bar{u}_1',\ldots,\bar{u}_m', \bar{w}_1,\ldots,\bar{w}_\ell)\}$. Then, by the Axiom of Extensionality, which is valid in the model, $v(z\in f)$ $=v(z\in f_1)$ for all $z\in D^S$. If $v(z\in\bar{w}_j)=v(z\in\bar{y}')$ for all $z\in D^S$, for some $\bar{y}'\in D^U$, then replace the $\bar{w}_j$ in A by the $\bar{y}'$. If there is no such $\bar{y}'\in D^U$ then replace any statement of the form $\bar{w}_j\in x$ in A by any false statement and replace any statement of the form $x'\in\bar{w}_j$ by its equivalent predicate expression, i.e. if $\bar{w}_j$ is $\{z' \ / \ B(z')\}$ then $x'\in\bar{w}_j$ is replaced by $B(x')$. For statements in A of the form $\bar{x}\in\bar{w}_j$, where $\bar{x}\in D^S-D^U$, if $v(z\in\bar{x})=v(z\in\bar{y}')$ for all $z'\in D^S$, for some $\bar{y}'\in D^U$ then replace $\bar{x}$ by $\bar{y}'$ and replace $\bar{y}'\in\bar{w}_j$ by its equivalent predicate expression, and if there is no such $\bar{y}'$ then replace $\bar{x}\in\bar{w}_j$ by any false statement. Let A' be the resulting form of A after these replacements have been made. Let g be $\{z' \ / \ A'(z', \bar{u}_1',\ldots,\bar{u}_m', \bar{x}_1',\ldots,\bar{x}_p')\}$. $g\in D^0$, $v(z\in g)=v(z\in f_1)$ for all $z\in D^S$ and hence $v(z\in g)=v(z\in f)$ for all $z\in D^S$.

Theorem 3.

The Axiom of Replacement (R) in the form : $Un(f) \supset (Sg')(Ax')(x' \in g' \equiv (Sy')(\langle y', x'\rangle \in f \ \& \ y' \in f'))$, is valid in the model.

Proof. Let $f \in D^S - D^U$ and let f be univocal. By lemma 3, there is a $g \in D^0$ such that $v(z \in f) = v(z \in g)$, for all $z \in D^S$. Let g be $\{z' \ / \ A(z', \bar{u}_1', \ldots, \bar{u}_m')\}$. So $\langle y', x'\rangle \in f \simeq (Sz')(T(Aw')(w' \in z' \stackrel{\prime}{\equiv} w' \in \langle y', x'\rangle) \ \& \ A(z', \bar{u}_1', \ldots, \bar{u}_m'))$ is valid in the model. Let the expression on the R.H.S. of the '$\simeq$' be called $B(y', x', \bar{u}_1', \ldots, \bar{u}_m')$. Since f is univocal, so is $B(y', x', \bar{u}_1', \ldots, \bar{u}_m')$. Hence, by Theorem 2, the Axiom R is valid in the model.

Since Axiom R implies Axiom S formally, Axiom S is valid in the model.

We will now test the validity of Axiom I (Axiom of Infinity). If $v(g' \in M_\alpha) = 1$ then $v(g' \cup \{g'\} \in M_{\alpha+1}) = 1$ since $g' \cup \{g'\}$ can be taken as $\{z' \in M_\alpha \cup \{k\} \ / \ z' \in g' \ v. (Aw')(w' \in M_\alpha \supset T(w' \in z' \stackrel{\prime}{\equiv} w' \in g')) \ \& \ T(k \in z' \stackrel{\prime}{\equiv} k \in g')\}$. Also $v(\{z' \in M_0 \cup \{k\} \ / \ \sim(k \in \{k\})\} \in M_1) = 1$. Hence the required f' can be taken as M_ω.

We will now test the validity of Axiom D, the Axiom of Regularity. Since f has at least one member, which is a member of D^U, let α be the least ordinal such that some member of f is a member of M_α. Let $v(g' \in f) = 1$ and $v(g' \in M_\alpha) = 1$. Then g' is either $M_{\alpha-1}$, $\{k\}$ (if $\alpha = 0$), or of the form $\{z' \in M_{\alpha-1} \cup \{k\} \ / \ A_{M_{\alpha-1} \cup \{k\}}(z', \bar{w}_1', \ldots, \bar{w}_\ell')\}$, where $v(\bar{w}_i' \in M_{\alpha-1}) = 1$ or $\bar{w}_i'$ is k, for all i. Hence any member of g' will be a member of $M_{\alpha-1}$ or be k. Hence there are no set members of g' that are members of f and the Axiom D is valid in the model.

We will now test the validity of Axiom C, the Axiom of Constructibility. Formally this is $(Af')(S\alpha)(f'\in M_{\alpha})$. Firstly we need to show that the ordinals defined according to the formal theory and interpreted in the model are in one-one correspondence with the M_{α}'s of the model, that is, with the ordinals used to set up the model. Before doing this, we need the following lemma :

Lemma 4.

' g' is an ordinal ' is absolute, i.e. if $\mathrm{Trans}_1(h')$ & $g'\in h'$ & (g' is an ordinal)$_{h'}$ then g' is an ordinal, where (g' is an ordinal)$_{h'}$ means that all the bound variables in ' g' is an ordinal ' are restricted to h'.

Proof. Absoluteness can be shown for $f'=g'$, $h'=\{f',g'\}$, $h'=\langle f', g'\rangle$, etc. as in Cohen, [3], p.94. However, in the definition of ' g' is an ordinal ', we need to replace ' EWeg' ' by ' ECong' '. This can be done as in [20], pp.35-36. Then the absoluteness of ' g' is an ordinal ' will follow.

We will now define the ordinals in the model. Let 0 be $\{z'\in M_0\cup\{k\} / \sim(k\in\{k\})\}$. Then $v(0\in M_1)=1$ and the smallest ordinal α such that $v(0\in M_{\alpha})=1$ is 1. Let α be defined in the model and let the smallest ordinal β such that $v(\alpha\in M_{\beta})=1$ be $\alpha+1$. Let $\alpha+1$ be $\{z'\in M_{\alpha+1}\cup\{k\} / z'\in\alpha\, v.(Aw')(w'\in M_{\alpha+1}\supset T(w'\in z' \doteq w'\in\alpha))$ & $T(k\in z' \doteq k\in\alpha)\}$. Clearly

$v(\alpha+1 \in M_{\alpha+2})=1$. If $v(\alpha+1 \in M_{\alpha+1})=1$ then $\alpha+1$ is either M_α or a subset of $M_\alpha \cup \{k\}$. Since $v(\alpha \in \alpha+1)=1$ then $v(\alpha \in M_\alpha)=1$, which is a contradiction. Hence $\alpha+2$ is the smallest ordinal such that $v(\alpha+1 \in M_{\alpha+2})=1$. Now let α be a limit ordinal and assume that for all $\beta<\alpha$, β is defined in the model such that the smallest ordinal γ such that $v(\beta \in M_\gamma)=1$ is $\beta+1$. Let α be $\{z' \in M_\alpha \cup \{k\} \;/\; (z' \text{ is an ordinal})_{M_\alpha \cup \{k\}}\}$. Since ' f' is an ordinal ' is absolute and $M_\alpha \cup \{k\}$ is transitive then α is the set of all ordinals in $M_\alpha \cup \{k\}$ and is hence the required limit ordinal. Clearly $v(\alpha \in M_{\alpha+1})=1$. If $v(\alpha \in M_\alpha)=1$, then $v(\alpha \in M_\beta)=1$ for some $\beta<\alpha$. Since $v(\beta \in \alpha)=1$ then $v(\beta \in M_\beta)=1$, which is a contradiction. Hence $\alpha+1$ is the smallest ordinal such that $v(\alpha \in M_{\alpha+1})=1$.

Hence all the ordinals can be defined in the model satisfying the properties of the ordinals and such that the smallest ordinal β such that $v(\alpha \in M_\beta)=1$ is $\alpha+1$, for all the ordinals α. Hence the ordinals α defined in the model are in one-one correspondence with the M_α's of the model.

To show the validity of Axiom C in the model, we must show that the M_α's of the formal theory, when interpreted in the model, have the same members as the M_α's of the model. This is shown by transfinite induction on the ordinals, the one-one correspondence above dispelling any ambiguity between the ordinals defined in the model and the ordinals used to construct the model.

Clearly M_0 of the model can be taken as a member of D^U with the same members as that of the formally defined M_0, interpreted in the model. Assume that the same holds for M_α. $M_{\alpha+1}$ is formally defined as the union of M_α and the set of all sets f' such that there is a predicate A, which is significant for all substitutions into its free variables, and $f' = \{z' \in M_\alpha \cup I \mid A_{M_\alpha \cup I}(z', \bar{w}_1', \ldots, \bar{w}_\ell')\}$, where $\bar{w}_i' \in M_\alpha \cup I$, for all i. Since $M_\alpha \cup \{k\}$ can be taken as $M_\alpha \cup I$, interpreted in the model, $\{z' \in M_\alpha \cup \{k\} \mid A_{M_\alpha \cup \{k\}}(z', \bar{w}_1', \ldots, \bar{w}_\ell')\}$ can be taken as $\{z' \in M_\alpha \cup I \mid A_{M_\alpha \cup I}(z', \bar{w}_1', \ldots, \bar{w}_\ell')\}$, interpreted in the model. Hence $M_{\alpha+1}$ of the model can be taken as the formal $M_{\alpha+1}$, interpreted in the model. If α is a limit ordinal and the above property holds for all $\beta < \alpha$, then the M_α of the model, satisfying the property of being the union of all the M_β's such that $\beta < \alpha$, can be taken as the formal M_α, interpreted in the model.

Since there is an α such that $v(f' \in M_\alpha) = 1$, for all $f' \in D^U$, $(Af')(S\alpha)(f' \in M_\alpha)$ is valid in the model. Hence Axiom C is valid in the model.

The next step is to show that the Axiom of Choice (A.C.) is valid in the model, using Axiom C. There are various equivalents of the Axiom of Choice, which can be shown by the methods in Mendelson, [17], pp.197-199, with little or no modification to allow for individuals. One of these equivalents is the Well-Ordering Principle : $(Af')(Sg')(g' We f')$, and so it is sufficient to prove it. This proof follows that in Cohen, [3], p.95.

Lemma 5.

There is a wff $A(h', j', M_\alpha \cup \{k\}, g')$ such that if g' is a well-ordering of the set $M_\alpha \cup \{k\}$, the relation $h' < j' \equiv A(h', j', M_\alpha \cup \{k\}, g')$ induces a well-ordering of the set $M_{\alpha+1} \cup \{k\}$, where A is significant for all substitutions into its free variables.

<u>Proof.</u> Enumerate the countably many formulae $B_n(x', t_1', \ldots, t_k')$. We have already essentially shown how to express the relation $C(h', n, t_1', \ldots, t_k') : h' = \{z' \in M_\alpha \cup \{k\} \ / \ (B_n)_{M_\alpha \cup \{k\}}(z', t_1', \ldots, t_k')\}$. Now the well-ordering g' induces a natural well-ordering on the set of all possible $(k+1)$-tuples $\langle n, t_1', \ldots, t_k' \rangle$ where $t_i' \in M_\alpha \cup \{k\}$, for all i. For each $h' \in M_{\alpha+1}$ we can define $\phi(h')$ as the first $(k+1)$-tuple, for some k, under this well-ordering, such that $C(h', \phi(h'), t_1', \ldots, t_k')$ holds. Now we can define A by having $h' < j'$ mean $\phi(h') < \phi(j')$. One can easily add k at the beginning of the well-ordering so that k is the first member of $M_{\alpha+1} \cup \{k\}$. Thus $M_{\alpha+1} \cup \{k\}$ can be well-ordered.

By transfinite induction, we can define a well-ordering on $M_\alpha \cup \{k\}$ as follows : $M_0 \cup \{k\}$ is $\{k, \{k\}\}$ and so can be well-ordered. If α is a limit ordinal and the well-ordering has been defined for all $M_\beta \cup \{k\}$ with $\beta < \alpha$, we well-order $M_\alpha \cup \{k\} = \bigcup_{\beta < \alpha} M_\beta \cup \{k\}$ in an obvious manner. By lemma 5, if $M_\alpha \cup \{k\}$ can be well-ordered then $M_{\alpha+1} \cup \{k\}$ can be well-ordered, and so $M_\alpha \cup \{k\}$ can be well-ordered for all α. Since Axiom C is valid in the model, let $\phi(f')$ be the least ordinal α such that $v(f' \in M_\alpha) = 1$. Define $f' < g'$ if $\phi(f') < \phi(g')$ or if $\phi(f') = \phi(g') = \alpha$ and f' precedes g' in the well-ordering of $M_\alpha \cup \{k\}$. Thus we have given a single formula $A(f', g')$ which well-orders all sets. Hence Axiom A.C. is valid in the model.

The next step is to show that Axiom GCH is valid in the model, using Axioms C and A.C. The proof follows that in Cohen,[3], pp. 95-98 and 82-83. Instead of using ranks in the Skolem-Lowenheim Theorem on p.82, use the least ordinal α such that $f' \in M_\alpha$. This does the required job of restricting the Axiom of Choice to sets and so the theorem follows similarly to the proof of the validity of the Axiom of Replacement in the model. One does, of course, only need to consider formulae $A(x_1', \ldots, x_n')$ containing only the connectives $\sim$, &, and T and with its quantifiers, A and E, at the beginning of the formula.

<u>Lemma 6.</u>

For all infinite α, $\overline{\overline{M_\alpha}} = \overline{\overline{\alpha}}$, in the model.

<u>Proof.</u> $\overline{\overline{M_n}}$ is finite, for all integers n. $\overline{\overline{M_n}} \geq n$ since $\alpha \in M_{\alpha+1}$, for all ordinals α. Hence $\overline{\overline{M_\omega}} = \aleph_0 = \overline{\overline{\omega}}$. If α is a successor ordinal and $\overline{\overline{M_\beta}} = \overline{\overline{\beta}}$ for all $\beta \leq \alpha - 1$, then the number of predicates $A_{M_{\alpha-1} \cup \{k\}}$ is $\overline{\overline{M_{\alpha-1}}}$ and hence $\overline{\overline{M_\alpha}} = \overline{\overline{M_{\alpha-1}}} = \overline{\overline{\alpha-1}} = \overline{\overline{\alpha}}$. If α is a limit ordinal and $\overline{\overline{M_\beta}} = \overline{\overline{\beta}}$ for all $\beta < \alpha$, then $\overline{\overline{M_\alpha}} = \overline{\overline{\bigcup_{\beta<\alpha} M_\beta}} \leq \overline{\overline{\alpha}} \times \overline{\overline{\alpha}} = \overline{\overline{\alpha}}$. Since $\beta \in M_{\beta+1}$, $\overline{\overline{M_\alpha}} \geq \overline{\overline{\alpha}}$ and hence $\overline{\overline{M_\alpha}} = \overline{\overline{\alpha}}$. Hence, for all infinite α, $\overline{\overline{M_\alpha}} = \overline{\overline{\alpha}}$ in the model.

Thus, in the model, lemma 1 of p.96, [3], follows, where a set f' is extensional if g' and $h' \in f'$ and $\sim(g'=h')$ implies $(Sx')(x' \in f'$ & $(x' \in g'$ & $\sim x' \in h')$ v $(x' \in h'$ & $\sim x' \in g'))$. In the theorem on p.73,

[3], concerning the unique one-one map ϕ from an extensional set to a transitive set, let the rank of f' be the least α such that $f' \in M_\alpha$, and if k is an individual $\in A$ then let $\phi(k)=k$ and if rank(f') =0 then let $\phi(f')=f'=\{k\}$. In the proof of ϕ being one-one, where $\alpha = \max(\text{rank } x, \text{rank } y)=0$, $x=y=\{k\}$, since the proof is being carried out in the model. If x and y are individuals then it is not the case that $\sim(x=y)$. If x is an individual and y is a set then $\sim T(\phi(x)=\phi(y))$. So the one-one condition is : $T(x=y)$ iff $T(\phi(x)=\phi(y))$. The rest of the proof follows as in Cohen, [3], and we can use the unique $\in$-isomorphism in the proof of Theorem 1 in pp. 95-97, [3]. The next result we need is the absoluteness of ' $x' \in M_\alpha$ '. Since the proof is being done in the model, if $k \in T$ and $l \circ k$ then $l \in T$ for any transitive set T. Hence we can show that ' $x'=y'$ ' is absolute and use this to show the absoluteness of ' $x' \in M_\alpha$ ', following through the steps in Cohen, [3], p.94, and using my formal definition of the M_α's. Now Theorem 1 (p.95-7, [3]) will follow. The Axiom GCH can now be shown to be valid in the model by the proof at the bottom of p.98, [3].

Hence all the axioms are valid in the model and the formal system is consistent relative to the theory needed to set up the model. NBG is sufficient to do this, the D_α's being sets of expressions and D^U, D^O, $D^1, \dots, D^n, \dots, D^S$, all being proper classes of expressions. These expressions are treated as individuals and sets and classes are formed from them, but nowhere is there a need for the postulate that there is an individual nor is there a need for restricting consideration to the domain of individuals for the purpose of theorising about them and hence requiring at least one individual for the quantification theory to come out right. So the consistency of NBG without individuals is sufficient to guarantee the consistency of the above theory. Since NBG is relatively consistent to Z-F, the above formal theory is relatively consistent to Z-F.

This leaves a number of questions unanswered. We have not proved formally that Axiom C implies Axiom AC. This however looks very doubtful because Axiom C does not say anything about the well-ordering of the set of all individuals, I. However, if an extra axiom, call it WOI, was added which ensured the possibility of well-ordering the set of all individuals, then it seems likely that Axiom AC would follow.

However, Axiom C formally implies Axiom D.

It also seems likely that Axiom C, together with Axiom WOI, formally implies Axiom GCH. To prove this, it seems, involves dispensing with the individuals altogether in the normal proof of Axiom GCH from Axiom C because they affect the cardinalities in the form of Axiom GCH. It seems the result can be proved by building up a transfinite sequence of N_α's, similar to the M_α's, but with $N_0=0$ instead of $\{\{k\},\{l\}, \text{etc.}\}$, and so the individuals are excluded completely from the construction. Then show that the

Axiom GCH holds for the sets belonging to the class $\underset{\alpha}{\mathrm{U}} N_{\alpha}$ and, using the Axiom AC, that to each set belonging to $\underset{\alpha}{\mathrm{U}} M_{\alpha}$ there is a set belonging to $\underset{\alpha}{\mathrm{U}} N_{\alpha}$ with the same cardinality.

However, Axiom GCH formally implies Axiom AC.

The question that now arises is that of the independence of the Axioms C, GCH and AC. It seems likely that these can be shown by forming <u>inner</u> models along similar lines to those of Cohen in [3]. As well as the "generic" sets that Cohen uses, one also needs the generic sets, $\{k\}$, $\{l\}$, etc., for all the individuals, and at each stage in the transfinite construction one should form subsets of $M_{\alpha} \cup I$, as in the model N of this chapter.

One could also add to the formal system ordinary language predicates so that these can be used to generate classes. The addition of these does not affect the consistency proof nor the development of the formal theory if they are introduced by adding general predicate variables and general subject variables to the formal theory. For the purpose of proving consistency one can specialise the predicate variables to those concerning membership and overlapping of special classes and individuals. In the development of the formal theory, whenever a wff-schema appears, as in forms of the Abstraction Axiom and Axiom of Replacement, a general predicate variable can be substituted. This would then allow one to generate classes from ordinary language predicates.

The theory of significance ranges for this formal theory will be developed in Chapter 8, where they will be considered on their own with the class theory used just to define them and show their properties.

CHAPTER 5.

A 3-VALUED CLASS THEORY AVOIDING THE PARADOXES.

In this chapter, I wish to present a 3-valued class theory with only the Axioms of Abstraction and Extensionality so that the class paradoxes can be avoided. This is the one mentioned in the Introduction. I use a 3-valued Lukasiewicz logic and restrict the connectives and quantifiers, used in constructing predicates for the Abstraction Axiom, to ~, & and A. I have explained how other such connectives lead to contradictions and have explained how the present theory avoids some of the class paradoxes. The formalisation is as follows:

Primitives

1. U, V, W, X, Y, Z, ____ (variables over classes.)
2. $\in$ (is a member of).
3. ~, $\rightarrow$, A (connectives and quantifier of the Lukasiewicz 3-valued logic).

Formation Rules.

1. If X and Y are variables then $X\in Y$ is an atomic wff.
2. The propositional constants 1, 0, $\frac{1}{2}$, are atomic wffs.
3. If B and C are wffs and X is a variable, then ~B, $B\rightarrow C$ and (AX)B are wffs.

Definitions.

$X=Y=df(AZ)(Z\in X\leftrightarrow Z\in Y)$= (X is identical with Y.)

$V(X)=df(AZ)C(Z\in X)$. (X is 2-valued.)

Axioms.

A. $(SY)(AX)(X\in Y\leftrightarrow\phi(X, Z_1, \_\_\_, Z_n))$, where ϕ is either a propositional constant or constructed from atomic wffs of the form $U\in V$ by using $\sim$, & and A.

E. $(AZ)(Z\in X\leftrightarrow Z\in Y)\supset(AW)(X\in W\leftrightarrow Y\in W)$.

Theorems.

T.1. $X=X$.

Defn. = : T.1.

T.2. $X=Y\ Y=X$.

Defn. = : T.2.

T.3. $X=Y\supset.Y=Z\supset X=Z$.

Defn. = : $X=Y\leftrightarrow(AW)(W\in X\leftrightarrow W\in Y)$___(1)

Defn. = : $Y=Z\leftrightarrow(AW)(W\in Y\leftrightarrow W\in Z)$___(2)

(1), (2) : $X=Y\supset.Y=Z\supset(AW)(W\in X\leftrightarrow W\in Z)$___(3)

(3), Defn. = : T.3.

T.4. $X=Y\supset.\phi(X)\leftrightarrow\phi(Y)$, for any wff ϕ

Defn. = : $X=Y\supset.Z\in X\leftrightarrow Z\in Y$___(1)

Defn. =, Ax.E : $X=Y\supset.X\in W\leftrightarrow Y\in W$___(2)

By using induction on the number of connectives and quantifiers in ϕ, T.4 can be shown.

T.5. $(S!Y)(AX)(X\in Y\leftrightarrow 0)$.

Ax.A : $(SY)(AX)(X\in Y\leftrightarrow 0)$___(1)

Hyp : $(AX)(X\in Y_1\leftrightarrow 0)$ & $(AX)(X\in Y_2\leftrightarrow 0)$___(2)

(2) : $(AX)X\in Y_1\leftrightarrow X\in Y_2)$___(3)

(3), Defn. = : $Y_1=Y_2$ ___ (4)

(2), (4) : $(AX)(X\epsilon Y_1 \leftrightarrow 0)$ & $(AX)(X\epsilon Y_2 \leftrightarrow 0) \supset Y_1=Y_2$ ___ (5)

(1), (5) : T.5.

[Note that $(S!Y)A(Y) =df (SY)(A(Y)$ & $(AZ)(A(Z) \supset Y=Z))$. Also, it can be shown that if $(S!Y)A(Y)$ is provable then one can introduce a new symbol into the theory for this unique Y without introducing any essentially new theorems into the theory because of such a symbol.]

Introduce the definition $\emptyset$ for the unique class Y such that $(AX)(X\epsilon Y \leftrightarrow 0)$, i.e. $(AX)F(X\epsilon\emptyset)$ holds.

T.6. $\underline{(S!Y)(AX)(X\epsilon Y \leftrightarrow 1)}$.

Ax.A : $(SY)(AX)(X\epsilon Y \leftrightarrow 1)$ ___ (1)

Hyp : $(AX)(X\epsilon Y_1 \leftrightarrow 1)$ & $(AX)(X\epsilon Y_2 \leftrightarrow 1)$ ___ (2)

(2) : $(AX)(X\epsilon Y_1 \leftrightarrow X\epsilon Y_2)$ ___ (3)

(3), Defn = : $Y_1=Y_2$ ___ (4)

(1), (2), (4) : T.6.

Introduce the definition U for the unique class Y such that $(AX)(X\epsilon Y \leftrightarrow 1)$, i.e. $(AX)T(X\epsilon U)$ holds.

T.7. $\underline{(S!Y)(AX)(X\epsilon Y \leftrightarrow \tfrac{1}{2})}$

Ax.A : $(SY)(AX)(X\epsilon Y \leftrightarrow \tfrac{1}{2})$ ___ (1)

Hyp : $(AX)(X\epsilon Y_1 \leftrightarrow \tfrac{1}{2})$ & $(AX)(X\epsilon Y_2 \leftrightarrow \tfrac{1}{2})$ ___ (2)

(2) : $(AX)(X\epsilon Y_1 \leftrightarrow X\epsilon Y_2)$ ___ (3)

(3), Defn. = : $Y_1=Y_2$ ___ (4)

(1), (2), (4) : T.7.

Introduce the definition H for the unique class Y such that $(AX)(X \in Y \leftrightarrow \frac{1}{2})$, i.e. $(AX)P(X \in H)$ holds.

T.8. $\underline{V(\emptyset)\ \&\ V(U)\ \&\ \sim V(H).}$

Defns. V, $\emptyset$, U, H : T.8.

T.9. $\underline{(S!Y)(AX)(X \in Y \leftrightarrow (AZ)(\sim Z \in X \ v\ Z \in W.\&.\sim Z \in W\ v\ Z \in X)).}$

Ax.A : $(SY)(AX)(X \in Y \leftrightarrow (AZ)(\sim Z \in X\ v\ Z \in W.\&.\sim Z \in W\ v\ Z \in X))$ ___(1)

Similarly to the above proofs, T.9 follows.

Introduce the definition $\{W\}$ for the unique class Y such that $(AX)(X \in Y \leftrightarrow (AZ)(\sim Z \in X\ v\ Z \in W.\&.\sim Z \in W\ v\ Z \in X))$.

T.10. $\underline{T(X \in \{W\}) \equiv T(X=W)\ \&\ V(W)}$

Defn. $\{W\}$: $T(X \in \{W\}) \equiv (AZ)(F(Z \in X)\ v\ T(Z \in W).\&.F(Z \in W)\ v\ T(Z \in X))$ ___(1)

Hyp : $T(X \in \{W\})$ ___(2)

Hyp : $\sim V(W)$ ___(3)

(3), Defn. V : $(SZ)P(Z \in W)$ ___(4)

(1), (2), (4) : $(SZ)(F(Z \in X)\ \&\ T(Z \in X))$ ___(5)

(3), (5) : $V(W)$ ___(6)

Similarly, $V(X)$ ___(7)

(1), (2), (6), (7) : $(AZ)(Z \in X \leftrightarrow Z \in W)$ ___(8)

(8), Defn. = : $T(X=W)$ ___(9)

(2), (6), (9) : $T(X \in \{W\}) \supset T(X=W)\ \&\ V(W)$ ___(10)

Hyp : $T(X=W)\ \&\ V(W)$ ___(11)

(11), T.4 : $V(X)$ ___(12)

(11), (12), Defn = : $(AZ)(F(Z \in X)\ v\ T(Z \in W).\&.F(Z \in W)\ v\ T(Z \in X))$ ___(13)

(1), (13) : $T(X\in\{W\})$ ___(14)

(11), (14) : $T(X=W)\ \&\ V(W)\ T(X\in\{W\})$ ___(15)

(10), (15) : T.10.

T.11 $\underline{F(X\in\{W\})\equiv(SZ)(T(Z\in X)\ \&\ F(Z\in W).v.F(Z\in X)\ \&\ T(Z\in W)).}$

Defn. $\{W\}$: $F(X\in\{W\})\equiv F(AZ)(\sim Z\in X \vee Z\in W.\&.\sim Z\in W \vee Z\in X)$ ___(1)

(1) : T.11.

T.12. $\underline{(AX)P(X\in\{H\}),\ \text{i.e.}\ \{H\}=H.}$

Defn. H : $P(Z\in H)$ ___(1)

T.8, T.10 : $\sim T(X\in\{H\})$ ___(2)

(1), T.11 : $\sim F(X\in\{H\})$ ___(3)

(2), (3) : T.12.

T.13. $\underline{(AX)P(H\in\{X\}).}$

Hyp : $T(H\in\{X\})$ ___(1)

(1), T.10 : $T(H=X)\ \&\ V(X)$ ___(2)

(2), T.4: $V(H)$ ___(3)

(1), (3), T.8 : $\sim T(H\in\{X\})$ ___(4)

Defn. H : $P(Z\in H)$ ___(5)

(5), T.11 : $\sim F(H\in\{X\})$ ___(6)

(4), (6) : T.13.

T.14. $\underline{T(X\in\{\emptyset\})\equiv T(X=\emptyset).\&.T(\emptyset\in\{X\})\equiv T(X=\emptyset).}$

T.8, T.10 : $T(X\in\{\emptyset\})\equiv T(X=\emptyset)$ ___ (1)

T.10 : $T(\emptyset\in\{X\})\equiv T(X=\emptyset)\ \&\ V(X)$ ___(2)

T.8, T.4 : $T(X=\emptyset)\supset V(X)$ ___(3)

(2), (3) : $T(\emptyset\in\{X\})\equiv T(X=\emptyset)$ ___(4)

(1), (4) : T.14.

T.15. $\underline{F(X\in\{\emptyset\})\equiv(SZ)T(Z\in X).\&.F(\emptyset\in\{X\})\equiv(SZ)T(Z\in X).}$

T.11, Defn $\emptyset$: T.15.

T.16. $\underline{T(X\in\{U\})\equiv T(X=U).\&.T(U\in\{X\})\equiv T(X=U).}$

T.8, T.10 : $T(X\in\{U\})\equiv T(X=U)$ ___(1)

T.10 : $T(U\in\{X\})\equiv T(X=U)\ \&\ V(X)$ ___(2)

T.8, T.4 : $T(X=U)\supset V(X)$ ___ (3)

(2), (3) : $T(U\in\{X\})\equiv T(X=U)$ ___(4)

(1), (4) : T.16.

T.17. $\underline{F(X\in\{U\})\equiv(SZ)F(Z\in X).\&.F(U\in\{X\})\equiv(SZ)F(Z\in X).}$

T.11, Defn U : T.17.

T.18. $\underline{\sim V(\{X\}).}$

T.13, Defn. V : T.18.

T.19 $\underline{\sim T(Y\in\{\{X\}\}).}$

T.10, T.18 : T.19.

T.20. $\underline{\sim T(\{Y\}\in\{X\})}$

T.4, T.10, T.18 : T.20

T.21. $\underline{V(X)\ \&\ \{X\}=\{Y\}\supset X=Y.}$

Hyp. : $V(X)\ \&\ \{X\}=\{Y\}$ ___(1)

(1), Defn. = : $(\Lambda Z)(Z\in\{X\}\leftrightarrow Z\in\{Y\})$ ___(2)

(2) : $X\in\{X\}\leftrightarrow X\in\{Y\}$___(3)

T.1, T.10, (1) : $X\in\{X\}$ ___(4)

(3), (4) : $X\in\{Y\}$ ___(5)

(5), T.10 : $T(X=Y)\ \&\ V(Y)$ ___ (6)

(6), : $X=Y$ ___ (7)

(1), (7) : T.21

T.22. $\underline{(S!Y)(AX)(X \in Y \leftrightarrow (AZ)(\sim Z \in X \vee Z \in V . \& . \sim Z \in V \vee Z \in X) \vee (AZ)(\sim Z \in X}$
$\underline{\vee Z \in W . \& . \sim Z \in W \vee Z \in X))}$.

Ax.A, Defn.=, as before : T.22.

Introduce the definition $\{V,W\}$ for the unique class Y such that $(AX)(X \in Y \leftrightarrow (AZ)(\sim Z \in X \vee Z \in V . \& . \sim Z \in V \vee Z \in X) \vee (AZ)(\sim Z \in X \vee Z \in W . \& . \sim Z \in W \vee Z \in X))$.

T.23. $\underline{T(X \in \{V,W\}) \equiv T(X=V) \ \& \ V(V) . \vee . T(X=W) \ \& \ V(W).}$

Defns. $\{U,V\}$, $\{V\}$, $\{W\}$: $X \in \{V,W\} \leftrightarrow X \in \{V\} \vee X \in \{W\}$ ______(1)

(1), T.10 : T.23.

T.24. $\underline{F(X \in \{V,W\}) \equiv F(X \in \{V\}) \ \& \ F(X \in \{W\}).}$

Defns. $\{V,W\}$, $\{V\}$, $\{W\}$, T.11 : T.24.

T.25. $\underline{P(H \in \{V,W\}).}$

T.23, T.8 : $\sim T(H \in \{V,W\})$ ______(1)

T.24, T.13 : $\sim F(H \in \{V,W\})$ ______(2)

(1), (2) : T.25.

T.26. $\underline{\sim V(\{V,W\}).}$

T.25, Defn. V : T.26.

T.27. $\underline{\sim T(X \in \{\{V,W\}\}).}$

T.10, T.26 : T.27.

T.28. $\underline{\sim T(\{V,W\} \in \{X\}).}$

T.4, T.10, T.26 : T.28.

T.29. $\underline{\sim T(\{V,W\} \in \{X\ ,Y\}).}$

T.4, T.26, T.23 : T.29.

T.30. $\underline{\sim T(X \in \{\{V\},\{V,W\}\}).}$

T.18, T.26, T.23 : T.30.

T.30 rules out the possibility of using the Kuratowski definition of ordered pair because it prevents a proof of $\{\{X\},\{X,Y\}\}=\{\{U\},\{U,V\}\} \supset X=Y \;\&\; U=V$.

Similarly to $\{X\}$ and $\{X,Y\}$, $\{X_1,\ldots,X_n\}$ can be defined so that $Z\in\{X_1,\ldots,X_n\} \leftrightarrow Z\in\{X_1\} \vee Z\in\{X_2\} \vee \ldots \vee Z\in\{X_n\}$. Using this definition, the following can be shown : $T(Z\in\{X_1,\ldots,X_n\}) \equiv V(Z) \;\&.\, T(Z=X_1) \vee \ldots \vee T(Z=X_n)$, $F(Z\in\{X_1,\ldots,X_n\}) \equiv F(Z\in\{X_1\}) \;\&\ldots\&\; F(Z\in\{X_n\})$, $P(H\in\{X_1,\ldots,X_n\})$, $\sim V(\{X_1,\ldots,X_n\})$, $\sim T(\{X_1,\ldots,X_n\}\in\{Y_1,\ldots,Y_m\})$, $\sim T(X\in\{\{Y_{1,1},\ldots,Y_{1,i_1}\},\ldots,\{Y_{n,1},\ldots,Y_{n,i_n}\}\})$. These rule out the use of classes of unordered n-tuples since they have no "members". Also unordered n-tuples cannot be "members" of any unordered m-tuple and so this restricts their use. This prevents a lot of the normal class-theoretic results from being proved. One could try to use an ordering relation to order an unordered pair as long as this relation is not a class of ordered pairs. In fact, even if one does form ordered pairs in this way one still cannot form useful classes of them to define relations and so their principal use is not there.

However, the ordered pair $\langle X,Y\rangle_R$ can be defined as $\{X,Y\}$ for a given wff $R(X,Y)$ such that $TR(X,Y) \;\&\; FR(Y,X)$ is provable. Assuming $V(X)$, $V(Y)$, $V(U)$ and $V(V)$, let $\langle X,Y\rangle_R = \langle U,V\rangle_S$. Then $\{X,Y\} = \{U,V\} \;\&\; TR(X,Y) \;\&\; FR(Y,X) \;\&\; TS(U,V) \;\&\; FS(V,U)$ holds. Hence $Z\in\{X\} \vee Z\in\{Y\} \leftrightarrow Z\in\{U\} \vee Z\in\{V\}$. Since $V(X)$, then $X\in\{X\}$ and $X\in\{U\} \vee X\in\{V\}$.

Hence $X=U \vee X=V$. Similarly, $Y=U \vee Y=V$. If $X=U$ then $\sim T(Y=U)$ and hence $Y=V$. Similarly, if $X=V$ then $Y=U$.

If $TR(U,V)$ & $FR(V,U)$ & $TS(U,V)$ & $FS(V,U)$ holds then call R and S <u>similar</u> relations.

If $TR(V,U)$ & $FR(U,V)$ & $TS(U,V)$ & $FS(V,U)$ holds then call R and S <u>dissimilar</u> relations. If R and S are similar then $X=U$ & $Y=V$ holds and if R and S are dissimilar then $X=V$ & $Y=U$ holds. Hence, under the assumption that $V(X)$, $V(Y)$, $V(U)$, $V(V)$ all hold and that R and S are similar then the usual property of ordered pairs : $\langle X,Y\rangle_R=\langle U,V\rangle_S \supset X=U$ & $Y=V$, holds.

The next question is whether one can successfully define classes of these ordered (or unordered) pairs as determined by a predicate. Let W_R be defined by : $Z\in W_R \leftrightarrow (SX)(SY)((AV)(\sim V\in Z \vee V\in\langle X,Y\rangle_R .\&. \sim V\in\langle X,Y\rangle_R \vee V\in Z)$ & $\phi(X,Y))$, where $TR(X,Y)$ & $FR(Y,X)$ holds for all X and Y. Then $T(Z\in W_R) \equiv (SX)(SY)((AV)(F(V\in Z) \vee T(V\in\langle X,Y\rangle_R) .\&. F(V\in\langle X,Y\rangle_R) \vee T(V\in Z))$ & $T\phi(X,Y))$. Since $P(H\in\{X,Y\})$, $P(H\in\langle X,Y\rangle_R)$ and hence $\sim T(Z\in W_R)$. Hence one cannot successfully define classes of ordered (or unordered) pairs as determined by a predicate. Hence, if one wants to deal with two-place relations, one cannot define them as classes of ordered pairs but must consider them as two-place predicates of the form : $\phi(X,Y)$. This prevents quantification over these relations. Many-place relations must also be considered in a similar manner because similar difficulties to the above ones for two-place relations will arise.

Another problem that arises in connection with unordered pairs is that $V(X)$ holds for so few classes X. The only classes so far established as satisfying this property are the universal class U and the null class $\emptyset$. It is not clear how any other classes satisfying $V(X)$ can be constructed.

In Whitehead and Russell's "Principia Mathematica", [34], p. 265 and p.357 (paperback edition), an ordered pair is defined as $\iota' x \uparrow \iota' y$, i.e. the class of all (ordered) couples (x,y) such that $x \in \iota' x$ & $y \in \iota' y$, where $\iota' x$ is the singleton of x. On p.200, they define a relation as a class of (ordered) couples such that a given function $\psi(x,y)$ is true. These definitions cannot be used in the above theory because there is nothing corresponding to classes of couples. One advantage that the type theory has is that it allows quantification over predicates and relations, which is one thing that cannot be done in the above theory.

However, let us go ahead with classes which can be defined from the Abstraction Axiom. The union $X \cup Y$ of two classes X and Y can be defined as follows : $(AZ)(Z \in X \cup Y \leftrightarrow Z \in X \vee Z \in Y)$. The intersection $X \cap Y$ of two classes X and Y can be defined as follows : $(AZ)(Z \in X \cap Y \leftrightarrow Z \in X \ \& \ Z \in Y)$.

T.31. $\underline{T(Z \in X \cup Y) \equiv T(Z \in X) \vee T(Z \in Y).}$

Defn. $\cup$: T.31.

T.32. $\underline{F(Z \in X \cup Y) \equiv F(Z \in X) \ \& \ F(Z \in Y).}$

Defn. $\cup$: T.32.

T.33. $\underline{T(Z \in X \cap Y) \equiv T(Z \in X) \ \& \ T(Z \in Y).}$

Defn. : T.33.

T.34. $\underline{F(Z \in X \cap Y) \equiv F(Z \in X) \vee F(Z \in Y).}$

Defn. : T.34.

The complement $\bar{X}$ of a class X can be defined as follows : (AZ) $(Z \in \bar{X} \leftrightarrow \sim Z \in X)$.

T.35. $\underline{T(Z \in \bar{X}) \equiv F(Z \in X) .\&. F(Z \in \bar{X}) \equiv T(Z \in X).}$

Defn.$^{-}$: T.35.

The domain $\mathfrak{D}(\phi)$ of a relation ϕ can be defined as follows : (AZ) $(Z \in \mathfrak{D}(\phi) \leftrightarrow (SV)\phi(Z,V))$. The range $\mathfrak{R}(\phi)$ of a relation ϕ can be defined as follows : $(AZ)(Z \in \mathfrak{R}(\phi) \leftrightarrow (SV)\phi(V,Z))$.

T.36. $\underline{T(Z \in \mathfrak{D}(\phi)) \equiv (SV)T\phi(Z,V).}$

Defn.$\mathfrak{D}$: T.36.

T.37. $\underline{F(Z \in \mathfrak{D}(\phi)) \equiv (AV)F\phi(Z,V).}$

Defn.$\mathfrak{D}$: T.37.

T.38. $\underline{T(Z \in \mathfrak{R}(\phi)) \equiv (SV)T\phi(V,Z).}$

Defn.$\mathfrak{R}$: T.38.

T.39. $\underline{F(Z \in \mathfrak{R}(\phi)) \equiv (AV)F\phi(V,Z).}$

Defn.$\mathfrak{R}$: T.39.

The following theorems follow immediately from the definitions.

T.40. $X \cap Y = Y \cap X$.

T.41. $X \cup Y = Y \cup X$.

T.42. $(X \cap Y) \cap Z = X \cap (Y \cap Z)$.

T.43. $(X \cup Y) \cup Z = X \cup (Y \cup Z)$.

T.44. $X \cap X = X$.

T.45. $X \cup X = X$.

T.46. $X \cap \emptyset = \emptyset$.

T.47. $X \cup \emptyset = X$.

T.48. $X \cap U = X$.

T.49. $X \cup U = U$.

T.50. $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$.

T.51. $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$.

T.52. $\overline{X \cup Y} = \overline{X} \cap \overline{Y}$.

T.53. $\overline{X \cap Y} = \overline{X} \cup \overline{Y}$.

T.54. $\overline{\overline{X}} = X$.

T.55. $\overline{\emptyset} = U$.

T.56. $\overline{U} = \emptyset$.

Define X-Y as $X \cap \overline{Y}$.

T.57. $T(Z \in X-Y) \equiv T(Z \in X)$ & $F(Z \in Y)$.

T.58. $F(Z \in X-Y) \equiv F(Z \in X)$ v $T(Z \in Y)$.

T.59. $U-X = \overline{X}$.

T.60. $\sim T(Z \in X \cap \overline{X})$.

T.61. $\sim T(Z \in X-X)$.

The Cartesian Product cannot be defined as it is a class of ordered pairs satisfying a predicate.

The Power Class $\mathcal{P}(X)$ of a class X can be defined as follows :

$(AZ)(Z \in \mathcal{P}(X) \leftrightarrow (AV)(\sim V \in Z \text{ v } V \in X))$.

T.62. $T(Z \in \mathcal{P}(X)) \equiv (AV)(F(V \in Z) \text{ v } T(V \in X))$.

Defn. $\mathcal{P}$: T.62.

T.63. <u>$F(Z \in \mathcal{P}(X)) \equiv (SV)(T(V \in Z) \ \& \ F(V \in X))$.</u>

Defn. $\mathcal{P}$: T.63.

T.64. <u>$\mathcal{P}(\emptyset) = \emptyset$.</u>

Defn. $\mathcal{P}$, $\emptyset$: $Z \in \mathcal{P}(\emptyset) \longleftrightarrow (AV)(\sim V \in Z)$ ______(1)

Defn. $\{\emptyset\}$: $Z \in \{\emptyset\} \longleftrightarrow (AV)(\sim V \in Z \ v \ V \in \emptyset . \& . \sim V \in \emptyset \ v \ V \in Z)$ ______(2)

(2) : $Z \in \{\emptyset\} \longleftrightarrow (AV)(\sim V \in Z)$ ______(3)

(1), (3), Defn.= : $\mathcal{P}(\emptyset) = \emptyset$.

T.65. <u>$\mathcal{P}(U) = U$.</u>

Defn. $\mathcal{P}$, U : $Z \in \mathcal{P}(U) \longleftrightarrow (AV)(\sim V \in Z \ v \ V \in U)$ ______(1)

(1) : $Z \in \mathcal{P}(U) \longleftrightarrow 1$ ______(2)

Defn. U : $Z \in U \longleftrightarrow 1$ ______(3)

(2), (3), Defn.= : $\mathcal{P}(U) = U$.

T.66. <u>$\sim T(X=U) \supset P(H \in \mathcal{P}(X))$.</u>

Hyp : $\sim T(X=U)$ ______(1)

Defn. $\mathcal{P}$: $H \in \mathcal{P}(X) \longleftrightarrow (AV)(\sim V \in H \ v \ V \in X)$ ______(2)

(2) : $\sim F(H \in \mathcal{P}(X))$ ______(3)

T.62 : $T(H \in \mathcal{P}(X)) \equiv (AV)T(V \in X)$ ______(4)

(1), (4) : $\sim T(H \in \mathcal{P}(X))$ ______(5)

(3), (5) : $P(H \in \mathcal{P}(X))$ ______(6)

(1), (6) : T.66.

T.67. <u>$U \in \mathcal{P}(X) \longleftrightarrow X=U$.</u>

Defn. $\mathcal{P}$: $U \in \mathcal{P}(X) \longleftrightarrow (AV)(\sim V \in U \ v \ V \in X)$ ______(1)

(1), Defn.U : $U \in \mathcal{P}(X) \longleftrightarrow (AV)(V \in X)$ ______(2)

Defn.= : $X=U \longleftrightarrow (AV)(V \in X \longleftrightarrow V \in U)$ ______(3)

(3), Defn.U : $X=U \leftrightarrow (AV)(V\in X)$ ____(4)

(2), (4) : $U\in \mathcal{P}(X) \leftrightarrow X=U$.

T.68. $\underline{\sim T(X=U) \supset \sim V(\mathcal{P}(X))}$.

T.66, Defn.V : T.68.

T.69. $\underline{T(\emptyset \in \mathcal{P}(X))}$.

Defn. $\mathcal{P}$: $Z\in \mathcal{P}(X) \leftrightarrow (AV)(\sim V\in Z \text{ v } V\in X)$ ______(1)

(1), Defn.$\emptyset$: $T(\emptyset \in \mathcal{P}(X))$.

T.70. $\underline{\sim F(X\in \mathcal{P}(X))}$.

Defn. $\mathcal{P}$: $X\in \mathcal{P}(X) \leftrightarrow (AV)(\sim V\in X \text{ v } V\in X)$ _____(1)

(1) : $\sim F(X\in \mathcal{P}(X))$.

T.71. $\underline{T(X\in \mathcal{P}(X)) \equiv V(X)}$.

Defn. $\mathcal{P}$: $T(X\in \mathcal{P}(X)) \equiv (AV)(F(V\in X) \text{ v } T(V\in X))$ _____(1)

(1), Defn.V : T.71.

T.72. $\underline{\sim T(X=U) \supset P(\mathcal{P}(X)\in \mathcal{P}(\mathcal{P}(X)))}$.

Hyp : $\sim T(X=U)$ ______(1)

(1), T.68, T.71 : $\sim T(\mathcal{P}(X)\in \mathcal{P}(\mathcal{P}(X)))$ ______(2)

T.70 : $\sim F(\mathcal{P}(X)\in \mathcal{P}(\mathcal{P}(X)))$ ______(3)

(2), (3) : $P(\mathcal{P}(X)\in \mathcal{P}(\mathcal{P}(X)))$ ______(4)

(1), (4) : T.72.

There are similar difficulties with the power classes as there are with the unordered n-tuples. These both arise because of the use of ~ and v in the Abstraction Axiom instead of the stronger $\rightarrow$ and $\leftrightarrow$, which lead to contradictions.

The Sum Class U(X) of a class X can be defined as follows :

$(AZ)(Z \in U(X) \leftrightarrow (SV)(Z \in V \ \& \ V \in X))$.

T.73. $\underline{T(Z \in U(X)) \equiv (SV)(T(Z \in V) \ \& \ T(V \in X))}$.

Defn.U : T.73.

T.74. $\underline{F(Z \in U(X)) \equiv (AV)(F(Z \in V) \ v \ F(V \in X))}$.

Defn.U : T.74.

T.75. $\underline{T(Z \in U(\{X,Y\})) \equiv T(Z \in X) \ \& \ V(X) . v . T(Z \in Y) \ \& \ V(Y)}$.

T.73, T.23 : $T(Z \in U(\{X,Y\})) \equiv (SV)(T(Z \in V) \ \&. (T(V=X) \ \& \ V(X)) \ v$

$(T(V=Y) \ \& \ V(Y)))$ _____(1)

(1) : $T(Z \in U(\{X,Y\})) \equiv . (SV)(T(Z \in V) \ \& \ T(V=X) \ \& \ V(X)) \ v \ (SV)(T(Z \in V)$

$\& \ T(V=Y) \ \& \ V(Y))$ _____(2)

(2) : T.75.

T.76. $\underline{\sim F(Z \in U(\{X,Y\}))}$.

Hyp : $F(Z \in U(\{X,Y\}))$ _____(1)

(1), T.74, T.24 : $(AV)(F(Z \in V) \ v . F(V \in \{X\}) \ \& \ F(V \in \{Y\}))$ _____(2)

(2) : $F(Z \in H) \ v . F(H \in \{X\}) \ \& \ F(H \in \{Y\})$ _____(3)

(1), (3), T.13, Defn.H : T.76.

Hence, if simple unions have to be formed it is better to avoid the construction $U(\{X_1, \dots, X_n\})$ for $X_1 \cup \dots \cup X_n$.

The Intersection Class $\bigcap(X)$ of a class X can be defined as follows :

$(AZ)(Z \in \bigcap(X) \leftrightarrow (AV)(\sim V \in X \ v \ Z \in V))$.

T.77. $\underline{T(Z \in \bigcap(X)) \equiv (AV)(F(V \in X) \ v \ T(Z \in V))}$.

Defn. $\bigcap$: T.77.

T.78. $\underline{F(Z \in \bigcap(X)) \equiv (SV)(T(V \in X) \ \& \ F(Z \in V))}$.

Defn. : T.78.

T.79. $\underline{\sim T(Z \in \bigcap(\{X,Y\}))}$.

T.77, T.24 : $T(Z \in \bigcap(\{X,Y\})) \equiv (AV)(F(V \in \{X\})\ \&\ F(V \in \{Y\})\ v.T(Z \in V))$ ___(1)

(1), Defn.H, T.13 : $\sim T(Z \in \bigcap(\{X,Y\}))$.

T.80. $\underline{F(Z \in \bigcap(\{X,Y\})) \equiv F(Z \in X)\ \&\ V(X).v.F(Z \in Y)\ \&\ V(Y)}$.

T.78, T.23 : $F(Z \in \bigcap(\{X,Y\})) \equiv (SV)((T(V=X)\ \&\ V(X).v.T(V=Y)\ \&\ V(Y))$
$\&\ F(Z \in V))$ ____(1)

(1) : $F(Z \in \bigcap(\{X,Y\})) \equiv (SV)(T(V=X)\ \&\ V(X)\ \&\ F(Z \in V))\ v\ (SV)(T(V=Y)$
$\&\ V(Y)\ \&\ F(Z \in V))$ _____(2)

(2) : T.80.

Hence, if simple intersections have to be formed it is better to avoid the construction $\bigcap(\{X_1,\ldots,X_n\})$ for $X_1 \cap \ldots \cap X_n$.

We will now give some definitions concerning relations.

If $\phi(X,Y) \leftrightarrow \psi(Y,X)$, for all X and Y, then ϕ and ψ are inverse relations of each other, ϕ is $\breve{\psi}$ and ψ is $\breve{\phi}$. Also $\mathfrak{R}(\phi) = \mathfrak{D}(\psi)$ and $\mathfrak{R}(\psi) = \mathfrak{D}(\phi)$.

$Un(\phi) =_{df} (AU)(AV)(AW)(\phi(U,V)\ \&\ \phi(U,W) \supset V=W)$. ($\phi$ is univocal.)

$Un_1(\phi) =_{df} Un(\phi)\ \&\ Un(\phi)$. ($\phi$ is one-one.)

$Z \upharpoonright \phi(X,Y) =_{df} X \in Z\ \&\ \phi(X,Y)$. ($\phi$ restricted to the domain Z.)

Note that $Z \upharpoonright \phi$ is a proposition, where $Y \upharpoonright X$, as in [17], p.168, is a class.

If there is a unique Z such that $\phi(Y,Z)$ then $Z =_{df} \phi{}^{\backprime}Y$; otherwise $\phi{}^{\backprime}Y =_{df} \emptyset$ (the null class).

$\phi{}^{\backprime\backprime}Y$ is defined as the unique W such that $(AZ)(Z \in W \leftrightarrow (SX)(Y \upharpoonright \phi(X,Z)))$.

$\phi{}^{\backprime\backprime}Y$ is the range of ϕ restricted to the domain Y.

The usual form of the Axiom of Infinity is : $(SX)(\emptyset\in X \;\&\; (AU)(U\in X \supset U\cup\{U\}\in X))$. The question arises as to whether $U\cup\{U\}$ is distinct from U for each U. Certainly $\emptyset$ and $\emptyset\cup\{\emptyset\}=\{\emptyset\}$ are distinct. $\{\emptyset\}$ and $\{\emptyset,\{\emptyset\}\}$ are distinct because :

T.81. $\underline{F(\{\emptyset\}\in\{\emptyset\})}$.

T.15 : $F(\{\emptyset\}\in\{\emptyset\}) \equiv (SZ)T(Z\in\{\emptyset\})$ ____(1)

T.14 : $T(\emptyset\in\{\emptyset\})$ ____(2)

(1), (2) : T.81.

T.82. $\underline{P(\{\emptyset\}\in\{\emptyset,\{\emptyset\}\})}$.

T.23 : $T(\{\emptyset\}\in\{\emptyset,\{\emptyset\}\}) \equiv T(\{\emptyset\}=\emptyset) \;\&\; V(\emptyset).v.T(\{\emptyset\}=\{\emptyset\}) \;\&\; V(\{\emptyset\})$ ____(1)

(1) : $\sim T(\{\emptyset\}\in\{\emptyset,\{\emptyset\}\})$ ____(2)

T.24 : $F(\{\emptyset\}\in\{\emptyset,\{\emptyset\}\}) \equiv F(\{\emptyset\}\in\{\emptyset\}) \;\&\; F(\{\emptyset\}\in\{\{\emptyset\}\})$ ____(3)

T.11. $F(\{\emptyset\}\in\{\emptyset\}) \equiv (SZ)(T(Z\in\{\emptyset\}) \;\&\; F(Z\in\{\emptyset\}) \;v\; (SZ)(F(Z\in\{\emptyset\}) \;\&\; T(Z\in\{\emptyset\}))$ ____(4)

(4), (3) : $\sim F(\{\emptyset\}\in\{\emptyset,\{\emptyset\}\})$ ____(5)

(2), (5) : T.82.

From here on, the position is not clear since the following are provable :

T.83. $T(\emptyset\in\{\emptyset,\{\emptyset\}\})$.

T.84. $P(\{\emptyset\}\in\{\emptyset,\{\emptyset\}\})$.

T.85. $P(\{\emptyset,\{\emptyset\}\}\in\{\emptyset,\{\emptyset\}\})$.

T.86. $T(\emptyset\in\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\})$.

T.87. $P(\{\emptyset\}\in\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\})$.

T.88. $P(\{\emptyset,\{\emptyset\}\}\in\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\})$.

T.89. $P(\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}\in\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\})$.

T.90. $P(\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}\in\{\emptyset,\{\emptyset\}\})$.

A similar situation arises if one tries to use power classes to generate an infinite number of classes. Clearly, $\emptyset$ and $\mathcal{P}(\emptyset)$ are distinct. Also $\mathcal{P}(\emptyset)$ and $\mathcal{P}(\mathcal{P}(\emptyset))$ are distinct because of the following :

T.91. $F(\mathcal{P}(\emptyset)\in\mathcal{P}(\emptyset))$.

T.92. $P(\mathcal{P}(\emptyset)\in\mathcal{P}(\mathcal{P}(\emptyset)))$.

But from here on, the position is not clear since the following are provable :

T.93. $T(\emptyset\in\mathcal{P}(\mathcal{P}(\emptyset)))$.

~~T.94. $P(\mathcal{P}(\emptyset)\in\mathcal{P}(\mathcal{P}(\emptyset)))$.~~

T.95. $P(\mathcal{P}(\mathcal{P}(\emptyset))\in\mathcal{P}(\mathcal{P}(\emptyset)))$.

T.96. $T(\emptyset\in\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))$.

T.97. $P(\mathcal{P}(\emptyset)\in\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))$.

T.98. $P(\mathcal{P}(\mathcal{P}(\emptyset))\in\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))$.

T.99. $P(\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))\in\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))))$.

T.100. $P(\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset)))\in\mathcal{P}(\mathcal{P}(\emptyset)))$.

So, the question of whether there are infinitely many classes remains open.

We will now give some more definitions concerning relations.

$\emptyset \mathrm{Irr} Y =_{df} (AZ)(Z\in Y \supset \sim T\emptyset(Z,Z))$. ($\emptyset$ is an irreflexive relation on Y.)

$\emptyset \mathrm{Tr} Y =_{df} (AU)(AV)(AW)(U\in Y \;\&\; V\in Y \;\&\; W\in Y \;\&\; \emptyset(U,V) \;\&\; \emptyset(V,W) \supset \emptyset(U,W))$.

($\emptyset$ is a transitive relation on Y.)

$\emptyset\mathrm{Part}Y =_{df} \emptyset\mathrm{Irr}Y \ \&\ \emptyset\mathrm{Tr}Y$. ($\emptyset$ partially orders Y.)

$\emptyset\mathrm{Con}Y =_{df} (AU)(AV)(U\in Y \ \&\ V\in Y \ \&\ {\sim}T(U=V) \supset. \emptyset(U,V) \vee \emptyset(V,U))$. ($\emptyset$ is a connected relation on Y.)

$\emptyset\mathrm{Tot}Y =_{df} \emptyset\mathrm{Irr}Y \ \&\ \emptyset\mathrm{Tr}Y \ \&\ \emptyset\mathrm{Con}Y$. ($\emptyset$ totally orders Y.)

$\emptyset\mathrm{We}Y =_{df} \emptyset\mathrm{Irr}Y \ \&\ (AZ)((AW)(W\in Z \supset W\in Y) \ \&\ (SX)(X\in Z) \supset (SU)(U\in Z \ \&\ (AV)(V\in Z \ \&\ {\sim}T(U=V) \supset T\emptyset(U,V) \ \&\ {\sim}T\emptyset(V,U))))$. ($\emptyset$ well-orders Y.)

$\mathrm{Sim}(\emptyset, f_1, X_1, f_2, X_2) =_{df} \mathrm{Un}_1(\emptyset) \ \&(AW)(W\in\mathfrak{D}(\emptyset) \equiv W\in X_1) \ \&\ (AW)(W \in \mathfrak{R}(\emptyset) \equiv W\in X_2) \ \&\ (AU)(AV)(U\in X_1 \ \&\ V\in X_1 \supset. f_1(U,V) \longleftrightarrow f_2(\emptyset{}^{\prime}U, \emptyset{}^{\prime}V))$.

($\emptyset$ is a similarity mapping of the relation f_1 on X_1 onto the relation f_2 on X_2.)

We cannot form the expression representing the existence of such a similarity mapping between relations on domains because it would require quantification over relations.

$\mathrm{Fld}(\emptyset) =_{df} \mathfrak{D}(\emptyset) \cup \mathfrak{R}(\emptyset)$. (The field of $\emptyset$.)

$\mathrm{TOR}(\emptyset) =_{df} \emptyset\mathrm{Tot}(\mathrm{Fld}(\emptyset))$. ($\emptyset$ is a total order.)

$\mathrm{WOR}(\emptyset) =_{df} \emptyset\mathrm{We}(\mathrm{Fld}(\emptyset))$. ($\emptyset$ is a well-ordering relation.)

T.101. $\underline{\mathrm{Sim}(\emptyset, f_1, X_1, f_2, X_2) \supset \mathrm{Sim}(\emptyset, f_2, X_2, f_1, X_1).}$

Defns. Sim, Un_1, $'$: T.101.

T.102. $\underline{\mathrm{Sim}(\emptyset, f_1, X_1, f_2, X_2) \ \&\ \mathrm{Sim}(\psi, f_2, X_2, f_3, X_3) \supset \mathrm{Sim}(\psi\circ\emptyset, f_1, X_1, f_3, X_3)}$, <u>where $\psi\circ\emptyset$ is the composition of $\emptyset$ and ψ.</u>

Defns. Sim, Un_1, $'$: T.102.

[$\psi\circ\emptyset(U,W) =_{df} (SV)(\emptyset(U,V) \ \&\ \psi(V,W))$.]

T.103. $\underline{\mathrm{Sim}(\emptyset, f_1, \mathrm{Fld}(f_1), f_2, \mathrm{Fld}(f_2)) \supset (\mathrm{TOR}(f_1) \equiv \mathrm{TOR}(f_2) .\&. \mathrm{WOR}(f_1) \equiv \mathrm{WOR}(f_2)).}$

Defns. Sim, Fld, Un_1, $'$: T.103.

The class of all total orders similar to X is called the order type of X. Order types cannot be formed because they require quantification over relations. This prevents the ordinals from being introduced by using order types. The paradigmatic method of introducing ordinals also fails because of what has already been shown about $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, etc. Similarly for cardinals, one cannot form the class of all classes which are equinumerous with a given class and also one cannot use any paradigmatic method of introducing them. However, one can express the fact that the relation ϕ yields a one-one correspondence between the members of the two classes X and Y.

$X \cong_{\phi} Y =_{df} Un_1(\phi)$ & $(AW)(W \in \mathfrak{D}(\phi) \equiv W \in X)$ & $(AW)(W \in \mathfrak{R}(\phi) \equiv W \in Y)$.

But one cannot form an expression for the fact that the members of the classes X and Y can be put in one-one correspondence.

T.104. $\underline{X \cong_{I} X, \text{ where } I(U,V) =_{df} U=V \ \& \ U \in X \ \& \ V \in X.}$

Defns. Un_1, I : $Un_1(I)$ ____(1)

Defns. $\mathfrak{D}$, I : $\mathfrak{D}(I)=X$ ____(2)

Defns. $\mathfrak{R}$, I : $\mathfrak{R}(I)=X$ ______(3)

(1), (2), (3), Defn. $\cong$: T.104.

T.105. $\underline{X \cong_{\phi} Y \supset Y \cong_{\breve{\phi}} X.}$

Defns. $\cong$, $\breve{\phi}$: T.105.

T.106. $\underline{X \cong_{\phi} Y \ \& \ Y \cong_{\psi} Z \supset X \cong_{\psi \circ \phi} Z.}$

Defns. $\cong$, o : T.106.

T.107. $X \underset{\phi}{\cong} Y \ \& \ X_1 \underset{\psi}{\cong} Y_1 \ \& \ X \cap X_1 = \emptyset \ \& \ Y \cap Y_1 = \emptyset \supset X \cup X_1 \underset{\phi \cup \psi}{\cong} Y \cup Y_1.$

Defn. $\cong$: T.107.

The class usually represented as X^Y cannot be formed as it is a class of functions.

$X \underset{\phi}{\lesssim} Y =_{df} (SZ)((AW)(W \in Z \supset W \in Y) \ \& \ X \underset{\phi}{\cong} Z)$. (The relation ϕ establishes a one-one correspondence between the members of X and the members of a sub-class of Y.)

When dealing with one-one correspondences there is only need to consider the members of the classes involved, i.e. Z is a member of Y iff $T(Z \in Y)$. Hence a subclass X of Y is taken to be such that all members of X are members of Y.

$X \prec Y$ cannot be formed in the usual way because one cannot express the fact that there are no one-one correspondences between X and Y.

T.108. $X \underset{I}{\lesssim} X.$

T.104, Defn. $\lesssim$: T.108.

T.109. $(AW)(W \in X \supset W \in Y) \supset X \underset{I}{\lesssim} Y$, where I is the identity relation on X.

T.104, Defn. $\lesssim$: T.109.

T.110. $X \underset{\phi}{\lesssim} Y \ \& \ Y \underset{\psi}{\lesssim} Z \supset X \underset{\psi \circ \phi}{\lesssim} Z.$

Hyp : $X \underset{\phi}{\lesssim} Y \ \& \ Y \underset{\psi}{\lesssim} Z$ ______(1)

(1), Defn. $\lesssim$: $(AW)(W \in Y_1 \supset W \in Y) \ \& \ X \underset{\phi}{\cong} Y_1$ ______(2) (Y_1 is a constant.)

(1), Defn. $\lesssim$: $(AW)(W \in Z_1 \supset W \in Z) \ \& \ Y \underset{\psi}{\cong} Z_1$ ______(3) (Z_1 is a constant.)

(2), (3) : $W \in \mathcal{R}(\psi o \phi) \supset W \in Z_1$ ______(4)

Defn.o, (3), (4) : $W \in \mathcal{R}(\psi o \phi) \supset W \in Z$ ______(5)

Defns. $\simeq$,o : $\mathcal{R}(\psi o \phi) \underset{\psi o \phi}{\simeq} X$ ____(6)

(5), (6), Defn. $\preceq$: $X \underset{\psi o \phi}{\preceq} Z$ ____(7)

(1), (7) : T.110.

I do not know whether the Schröder-Bernstein Theorem : $X \underset{\phi}{\preceq} Y$ & $Y \underset{\psi}{\preceq} X \supset X \underset{\chi}{\simeq} Y$, for some relation χ, follows or not, The type of proof given in Mendelson, [17], will not work. Cantor's Theorem, $X \prec \mathcal{P}(X)$, cannot be stated and, anyway, $U = \mathcal{P}(U)$ provides a counter-example to it.

As stated before, cardinal numbers cannot be introduced as such and the theory cannot usefully be carried much further.

As for the following forms of the Axiom of Choice, the Well-ordering Principle, Trichotomy, and the form asserting the existence of a choice function for any non-empty set, all cannot be stated. However we can state the Multiplicative Axiom as follows : $(AU)(U \in X \supset (SW)(W \in U) \ \& \ (AV)(V \in X \ \& \ {\sim}T(V=U) \supset V \cap U = \phi)) \supset (SY)(AU)(U \in X \supset (S!W)(W \in U \cap Y))$.

This would almost certainly be independent of the Axioms of Abstraction and Extensionality as there is no obvious predicate to generate the class Y. Zorn's Lemma can also be stated, i.e. ϕPartX & $(AU)((AW)(W \in U \supset W \in X) \ \& \ \phi\mathrm{Tot}U \supset (SV)(V \in X \ \& \ (AW)(W \in U \supset W = V \vee \phi(W,V)))) \supset (SV)(V \in X \ \& \ (AW)(W \in X \supset {\sim}T\phi(V,W)))$.

I do not know whether these two forms of the Axiom of Choice are equivalent in this theory. Many of the usual methods of proof would break down as they depend on ordered pairs, well-ordering, Cartesian Products, quantification over relations, and the like.

The last thing I wish to consider is the Axiom of Regularity. The form of it is as follows : $(AX)((SW)(W\in X) \supset (SY)(Y\in X \;\&\; Y\cap X=\emptyset))$. Let X be $\{U\}$. Then $Z\in\{U\} \leftrightarrow Z=U$ and $(SW)(W\in\{U\})$. $(SY)(Y\in\{U\} \;\&\; Y\cap\{U\}=\emptyset) \leftrightarrow (SY)(Y=U \;\&\; Y\cap\{U\}=\emptyset) \leftrightarrow U\cap\{U\}=\emptyset$. Since $U\cap\{U\}=\{U\}$, $F(U\cap\{U\}=\emptyset)$, $F((SY)(Y\in\{U\} \;\&\; Y\cap\{U\}=\emptyset))$ and hence $F(AX)((SW)(W\in X) \supset (SY)(Y\in X \;\&\; Y\cap X=\emptyset))$ The class $\{U\}$ falsifies the above Axiom of Regularity.

So there are many developments of ordinary set theory which do not follow for this theory. The Boolean operations, except for $X-X=\emptyset$, come out as usual, but there are few areas of ordinary set theory left unscathed. The theory, by itself, is rather weak. However, there are a number of possibilities for strengthening the theory.

If the predicate $\emptyset(X, Z_1,\ldots,Z_n)$ of the Abstraction Axiom was allowed to be constructed from atomic wffs of the form X=Y as well as $X\in Y$ then the theory about ordered and unordered pairs would turn out much more satisfactorily. As yet, I have not been able to establish any contradiction arising from its inclusion. But the consistency proof to follow would not be able to be used if X=Y was added. In fact, the method of proof would have to be changed radically to make allowances for the addition of X=Y. Also I have not been able to discover such a proof.

Another way of strengthening the theory would be to use a second order Lukasiewicz predicate logic instead of the first order one used. This would only be a minor strengthening compared to the previous suggestion but it would allow quantification over relations and functions. One could then express the statement that X is equinumerous with Y, the statement that for every non-empty class X there is a choice function for X, the statement that X has a lower cardinality than Y, and others.

One of the problems with the theory is that it is not strong enough to develop Mathematics. However this will be overcome in the next chapter by adding the class theory NBG (as in [17]) as a 2-valued sub-theory, to the above theory.

The remaining task of this chapter is to show that the above theory is relatively consistent to Z-F (as in [30]). Th. Skolem has produced models, in [26] and in [27] for an Abstraction Axiom the same as above except that $\emptyset$ may not be constructed using quantifiers A and S. He shows that the Axiom of Extensionality is valid also in his model in [27]. The procedure we use for constructing the model roughly follows the lines of P.C. Gilmore's paper, [8], where he constructed a model for his partial set theory PST'.

To construct the model, we need to extend the wffs used in the above theory by adding some terms, some of which will be used as the domain of the model. We give the formation rules for terms and wffs as follows :

1. If X and Y are class variables, then $X \in Y$ is an atomic wff.

2. Any combination of wffs using $\sim, \rightarrow$, A are wffs.

3. A propositional constant (i.e. 1, $\frac{1}{2}$ or 0) is an atomic wff.

4. A propositional constant (i.e. 1, $\frac{1}{2}$ or 0) or a wff constructed from atomic wffs using only $\sim$, &, A is a standard wff.

5. If P is a standard wff and X is a class variable, then $\{X : P\}$ is a term.

6. If $\{X : P\}$ and $\{X : Q\}$ are terms and Y is a class variable, then $\{X : P\} \in Y$, $Y \in \{X : P\}$, $\{X : P\} \in \{X : Q\}$ are atomic wffs.

We will use A, B, C, etc. for constant terms. We construct a model for the Abstraction Axiom with domain the set D of all constant terms $\{X : P\}$, i.e. P either has no free variables at all or has X as its only free variable. Non-constant terms can be defined from these as follows : Associate with any term $\{X : P(X, Z_1, \ldots, Z_k)\}$, for which $Z_1, \ldots, Z_k$ are the only free variables of the term, the function which for constant terms $A_1, \ldots, A_k$ of D takes as value the constant term $\{X : P(X, A_1, \ldots, A_k)\}$ of D.

Let any specification of values for all the constant atomic wffs of the form $X \in Y$, where X and Y range over the domain D, be called a <u>structure on D.</u> Let $v_M(P)$ denote the value of the constant wff P given by the structure M on D. Also let $v_M(1)=1$, $v_M(0)=0$ and $v_M(\frac{1}{2})=\frac{1}{2}$. Define $M_1 \leq M_2$ for two structures M_1 and M_2 on D if, for every constant atomic wff P, if $v_{M_1}(P)=1$ then $v_{M_2}(P)=1$

and if $v_{M_1}(P)=0$ then $v_{M_2}(P)=0$. Define the structure M_0, such that, for all constant atomic wffs P except propositional constants, $v_{M_0}(P)=\frac{1}{2}$. Then $M_0 \leq M$, for any structure M on D. '$\leq$' defines a partial ordering on the set of structures, since (i) $M \leq M$, (ii) if $M_1 \leq M_2$ and $M_2 \leq M_3$ then $M_1 \leq M_3$ and (iii) if $M_1 \leq M_2$ and $M_2 \leq M_1$ then $M_1 = M_2$ (i.e. M_1 and M_2 are the same structure.).

From now on, when mentioning values of wffs in a structure it is automatically assumed that the wffs are constant ones, i.e. they have no free variables.

Lemma 1.

Let M and M' be two structures on D such that $M \leq M'$. Then, for any standard wff P, if $v_M(P)=1$ then $v_{M'}(P)=1$ and if $v_M(P)=0$ then $v_{M'}(P)=0$.

Proof. By induction on wff evaluation procedure. This means that we start at the values of the substitution instances of all the atomic wffs and build up the value of P from these values according to the connectives and quantifiers in the Lukasiewicz logic. If P is an atomic wff or a propositional constant, the lemma holds.

(i) Let $v_M(\sim Q)=1$. Then $v_M(Q)=0$. By ind. hyp., $v_{M'}(Q)=0$. Hence $v_{M'}(\sim Q)=1$. Let $v_M(\sim Q)=0$. Then $v_M(Q)=1$. By ind. hyp., $v_{M'}(Q)=1$. Hence $v_{M'}(\sim Q)=0$.

(ii) Let $v_M(Q \& R)=1$. Then $v_M(Q)=1$ and $v_M(R)=1$. By ind. hyp., $v_{M'}(Q)=1$ and $v_{M'}(R)=1$. Hence $v_{M'}(Q \& R)=1$. Let $v_M(Q \& R)=0$. Then $v_M(Q)=0$ or $v_M(R)=0$. By ind. hyp., $v_{M'}(Q)=0$ or $v_{M'}(R)=0$. Hence

$v_{M'}(Q \& R)=0$.

(iii) Let $v_M((AX)Q)=1$. Then $v_M(Q(X))=1$ for all $X\in D$. By ind. hyp., $v_{M''}(Q(X))=1$ for all $X\in D$. Hence $v_{M'}((AX)Q)=1$. Let $v_M((AX)Q)=0$. Then $v_M(Q(X))=0$ for some $X\in D$. By ind. hyp., $v_{M'}(Q(X))=0$ for some $X\in D$. Hence $v_{M'}((AX)Q)=0$.

The model is the limit of a sequence of structures, $M_0 \leq M_1 \leq M_2 \leq \ldots \leq M_\mu \leq \ldots$, on D. M_0 is defined above, i.e. $v_{M_0}(P)=\frac{1}{2}$ for all atomic wffs P, except propositional constants. Assuming M_μ defined for some ordinal μ, $M_{\mu+1}$ is defined as follows : For all standard wffs P, $v_{M_{\mu+1}}(A\in\{X : P(X)\})=v_{M_\mu}(P(A))$.
For a limit ordinal μ, for all atomic wffs P, if $v_{M_\nu}(P)=1$ for some $\nu<\mu$ then $v_{M_\mu}(P)=1$, if $v_{M_\nu}(P)=0$ for some $\nu<\mu$ then $v_{M_\mu}(P)=0$, and if $v_{M_\nu}(P)=\frac{1}{2}$ for all $\nu<\mu$ then $v_{M_\mu}(P)=\frac{1}{2}$.

In the definition of M_μ for a limit ordinal μ, it was assumed that if $v_{M_\nu}(P)=1$ (or 0) for some $\nu<\mu$, then $v_{M_\tau}(P)=1$ (or 0) for all τ such that $\nu\leq\tau<\mu$. The construction of M_μ needs to be coupled with Lemma 2 so that when M_μ is formed the assumption above will be satisfied. That is, Lemma 2 is proved for each structure M_μ as it is constructed.

Before proving Lemma 2, I will give some examples in M_1, M_2 and M_3. Since standard wffs include propositional constants 0 and 1, by definition of M_1, $v_{M_1}(A\in\{X : 1\})=1$ and $v_{M_1}(A\in\{X : 0\})=0$. Let $\{X : 1\}$ be called U and $\{X : 0\}$ be called $\emptyset$. Hence $v_{M_1}(\emptyset\in U)=1$

and $v_{M_1}(U\in U)=1$. Using these two we can construct wffs taking values 1 or 0 in M_2. For example, $v_{M_2}(U\in\{X : \emptyset\in X\})=1$, $v_{M_2}(\emptyset\in\{X : \sim X\in X\})=1$, $v_{M_2}(U\in\{X : X\notin X\})=1$, $v_{M_2}(\emptyset\in\{X : \sim U\in X\})=1$. Let $\{C\}$ be $\{X : (AY)(\sim Y\in X \vee Y\in C.\&.\sim Y\in C \vee Y\in X)\}$. Then $v_{M_2}(\emptyset\in\{\emptyset\})=1$, $v_{M_2}(U\in\{U\})=1$, $v_{M_2}(U\in\{\emptyset\})=0$ and $v_{M_2}(\emptyset\in\{U\})=0$. Some examples in M_3 are the following : $v_{M_3}(\{\emptyset\}\in\{X : \emptyset\in X\})=1$, $v_{M_3}(\{U\}\in\{X : U\in X\})=1$, $v_{M_3}(\{X : \emptyset\in X\}\in\{X : U\in X\})=1$ and $v_{M_3}(\{\emptyset\}\in\{X : \sim X\in X\})=1$.

Lemma 2.

$M_\nu\leq M_\mu$, for all $\nu\leq\mu$.

Proof. By transfinite induction on μ. The ind. hyp. is $M_\nu\leq M_\tau$ for all $\nu\leq\tau$, for all $\tau<\mu$.

(i) $\mu=0$. M_0 M_0.

(ii) μ is a successor ordinal.

Let $v_{M_\nu}(A\in\{X : P\})=1$. There is a $\eta<\nu$ such that $v_{M_\eta}(P(A))=1$ by the method of construction of the structures. Since $\eta\leq\mu-1$, $M_\eta\leq M_{\mu-1}$ by the ind. hyp. Hence $v_{M_{\mu-1}}(P(A))=1$. By the construction of M_μ, $v_{M_\mu}(A\in\{X : P\})=1$. Similarly, if $v_{M_\nu}(A\in\{X : P\})=0$, then $v_{M_\mu}(A\in\{X : P\})=0$.

(iii) μ is a limit ordinal.

Let $\nu<\mu$. Let $v_{M_\nu}(A\in\{X : P\})=1$. Then $v_{M_\mu}(A\in\{X : P\})=1$, by definition of M_μ. Let $v_{M_\nu}(A\in\{X : P\})=0$. Then $v_{M_\mu}(A\in\{X : P\})=0$, by definition of M_μ.

Let $\nu=\mu$. Then $M_\nu\leq M_\mu$.

Lemma 3.

There is an ordinal λ of the second number class such that $M_\lambda = M_{\lambda+1}$.

Proof. The increasing chain of structures $M_0 \leq M_1 \leq \ldots\ldots \leq M_\mu \leq \ldots$ can be regarded as two increasing chains of subsets of the denumerable set of all atomic wffs of the form $A \in B$. One chain is of those atomic wffs taking the value 1 and the other is of those taking the value 0. If $M_\nu = M_{\nu+1}$, then $M_\nu = M_\mu$ for all ordinals μ such that $\nu \leq \mu$, since, by the method of construction, there is no way of changing the values of any atomic wffs. There is a denumerable set of ordinals μ such that $M_\mu \neq M_{\mu+1}$. But the set of all ordinals of the second number class is non-denumerable and hence for some λ in this class, $M_\lambda = M_{\lambda+1}$.

Theorem 1.

$V \in \{X : P\} \leftrightarrow P(V)$ is valid in M_λ, for all standard wffs P.

Proof. Let $v_{M_\lambda}(A \in \{X : P\}) = 1$. Let ν be the least ordinal such that $v_{M_\nu}(A \in \{X : P\}) = 1$. ν is a successor ordinal. Hence $v_{M_{\nu-1}}(P(A)) = 1$. Since $\nu - 1 \leq \lambda$, $M_{\nu-1} \leq M_\lambda$, by lemma 2. Hence $v_{M_\lambda}(P(A)) = 1$ since P is standard, by lemma 1. Similarly, if $v_{M_\lambda}(A \in \{X : P\}) = 0$, then $v_{M_\lambda}(P(A)) = 0$.

Let $v_{M_\lambda}(P(A)) = 1$. Then $v_{M_{\lambda+1}}(A \in \{X : P\}) = 1$. Since $M_\lambda = M_{\lambda+1}$, $v_{M_\lambda}(A \in \{X : P\}) = 1$. Similarly, if $v_{M_\lambda}(P(A)) = 0$, then $v_{M_\lambda}(A \in \{X : P\}) = 0$.

Theorem 2.

The Abstraction Axiom is valid in M_λ.

__Proof.__ By Theorem 1, for any standard wff P, $V\in\{X : P\}\leftrightarrow P(V)$ is valid in M_λ. Hence, $(SY)(AX)(X\in Y\leftrightarrow P(X, Z_1,\dots,Z_n))$ is valid in M_λ, for all wffs P which are propositional constants or constructed from atomic wffs of the form $X\in Y$ using only ~, & and A.

The next task is to prove that the Axiom of Extensionality is valid in M_λ. Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. Let ν_P be the least ordinal such that $v_{M_{\nu_P}}(P)=1$ or 0. Form the set of all substitution instances of all the atomic wffs of P which take the value 1 or 0 in M_{ν_P}. Call this __the dependent set of P, D(P).__

__Lemma 4.__

Let P(A) be a standard wff such that $v_{M_\lambda}(P(A))=1$ or 0. If, for each $Q(A)\in D(P(A))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$, then $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

__Proof.__ By induction on wff evaluation procedure. Let P(A) be an atomic wff such that $v_{M_\lambda}(P(A))=1$ or 0. Then $D(P(A))=\{P(A)\}$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(i) Let P(A) be ~R(A). Since $D(\sim R(A))=D(R(A))$, for each $Q(A)\in D(R(A))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp. $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(ii) Let P(A) be R(A) & S(A) and $v_{M_\lambda}(R(A)\ \&\ S(A))=1$. Then $v_{M_\lambda}(R(A))=1$ and $v_{M_\lambda}(S(A))=1$. Since $\nu_{R(A)}\leq\nu_{R(A)\ \&\ S(A)}$, $D(R(A))\subseteq D(R(A)\ \&\ S(A))$. Hence, for each $Q(A)\in D(R(A))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))$. Similarly, $v_{M_\lambda}(S(B))=v_{M_\lambda}(S(A))$.

Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iii) Let P(A) be R(A) & S(A) and $v_{M_\lambda}(R(A) \& S(A))=0$. Then $v_{M_\lambda}(R(A))=0$ or $v_{M_\lambda}(S(A))=0$. Since $\nu_{R(A)} = \nu_{R(A) \& S(A)}$ or $\nu_{S(A)} = \nu_{R(A) \& S(A)}$, $D(R(A)) \subseteq D(R(A) \& S(A))$ or $D(S(A)) \subseteq D(R(A) \& S(A))$. Hence, as above, $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))=0$ or $v_{M_\lambda}(S(B))=v_{M_\lambda}(S(A))=0$. Hence $v_{M_\lambda}(R(B) \& S(B))=0$ and $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iv) Let P(A) be (AZ)R(A,Z) and $v_{M_\lambda}((AZ)R(A,Z))=1$. Then $v_{M_\lambda}(R(A,Z))=1$, for all $Z \in D$. Since $\nu_{R(A,Z)} \leq \nu_{(AZ)R(A,Z)}$ for all Z, then $D(R(A,Z)) \subseteq D((AZ)R(A,Z))$ for all Z. Hence, for each $Q(A) \in D(R(A,Z))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B,Z))=v_{M_\lambda}(R(A,Z))$. Since this holds for all Z, $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(v) Let P(A) be (AZ)R(A,Z) and $v_{M_\lambda}((AZ)R(A,Z))=0$. Then $v_{M_\lambda}(R(A,Z))=0$ for some Z. Since $\nu_{R(A,Z)} = \nu_{(AZ)R(A,Z)}$ for some Z, say Z_0, then $D(R(A,Z_0)) \subseteq D((AZ)R(A,Z))$. For each $Q(A) \in D(R(A,Z_0))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B,Z_0))=v_{M_\lambda}(R(A,Z_0))=0$. Hence $v_{M_\lambda}((AZ)R(B,Z))=0$ and $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

Let P be an atomic wff (not 1 or 0) such that $v_{M_\lambda}(P)=1$ or 0. Define <u>the corresponding standard wff of P, C_P,</u> as follows : Let P have the form $A \in \{X : Q(X)\}$. Then C_P is Q(A).

Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. Let P have dependent set, D(P). We define a <u>general dependent set of P</u> as follows :

(i) The dependent set D(P) of P is a gen. dep. set of P.

(ii) If $v_{M_\lambda}(R)=1$ or 0 and R is an atomic wff (not 1 or 0), then $D(C_R)$ is a gen. dep. set of R.

(iii) Let D' be a gen. dep. set of P. Let $S \subseteq D'$. If $Q \in S$, let D_Q be a gen. dep. set of Q. Then $(D' \cap \bar{S}) \cup \bigcup_{Q \in S} D_Q$ is a gen. dep. set of P.

This assumes $v_{M_\lambda}(Q)=1$ or 0, for all $Q \in S$. Note that lemma 5 should be coupled with the definition of a gen. dep. set so that the assumption can be made before the construction of the gen. dep. sets D_Q.

Lemma 5.

Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. If D' is a gen. dep. set of P then, for each $Q \in D'$, $v_{M_\lambda}(Q)=1$ or 0.

Proof. By induction on the stages of construction of gen. dep. sets of all standard wffs such that $v_{M_\lambda}(P)=1$ or 0.

(i) By definition of D(P), if $Q \in D(P)$ then $v_{M_\lambda}(Q)=1$ or 0.

(ii) If $Q \in D(C_R)$, where R is an atomic wff (not 1 or 0) and $v_{M_\lambda}(R)$ =1 or 0, then $v_{M_\lambda}(Q)=1$ or 0.

(iii) Let D' be a gen. dep. set of P. Let $S \subseteq D'$. If $Q \in S$ then, by ind. hyp. for D', $v_{M_\lambda}(Q)=1$ or 0 and so, let D_Q be a gen. dep. set of Q. Let $T \in (D' \cap \bar{S}) \cup \bigcup_{Q \in S} D_Q$. If $T \in D_Q$, for some $Q \in S$, then by ind. hyp. for D_Q, $v_{M_\lambda}(T)=1$ or 0. If $T \in D' \cap \bar{S}$, then, by ind. hyp. for D', $v_{M_\lambda}(T)=1$ or 0. Hence, if $T \in (D' \cap \bar{S}) \cup \bigcup_{Q \in S} D_Q$, then $v_{M_\lambda}(T)=1$ or 0.

Lemma 6.

Let P be an atomic wff such that $v_{M_\lambda}(P)=1$ or 0. If D' is a gen.

dep. set of P which is not D(P) then, for each $Q \in D'$, $v_{M_{\nu_P - 1}}(Q)$ =1 or 0.

Proof. By transfinite induction on the ordinals ν_P. The ind. hyp. is that the lemma holds for all atomic wffs Q such that $\nu_Q < \nu_P$.

(i) $\nu_P = 0$. P is 1 or 0. The only gen. dep. set of P is of the form D(P). Hence the lemma holds vacuously.

(ii) ν_P is a successor ordinal. Use induction on the stages of construction of gen. dep. sets of P.

(I) D(P) is not used as a gen. dep. set in this lemma.

(II) If $v_{M_\lambda}(R)=1$ or 0, R is an atomic wff (not 1 or 0) and if $Q \in D(C_R)$, then $v_{M_{\nu_R - 1}}(Q)=1$ or 0. In the process of construction of gen. dep. sets of P, R is either P itself or is a member of a gen. dep. set of P. If R is P itself, then $v_{M_{\nu_P - 1}}(Q)=1$ or 0. If R is a member of a gen. dep. set of P, then, by ind. hyp., $v_{M_{\nu_P - 1}}(R)=1$ or 0 or $v_{M_{\nu_P}}(R)=1$ or 0, the latter being the case when R is a member of the dependent set of P. Hence $\nu_R \leq \nu_P$ and if $Q \in D(C_R)$ then $v_{M_{\nu_P - 1}}(Q)=1$ or 0.

(III) Let D' be a gen. dep. set of P for which the lemma holds. Let $S \subseteq D'$. If $Q \in S$, let D_Q be a gen. dep. set of Q. By ind. hyp. for D', $v_{M_{\nu_P - 1}}(Q)=1$ or 0, for all $Q \in S$. By ind. hyp. for the ordinals, the lemma holds for any gen. dep. set of Q except for D(Q).

Let $T\in(D'\cap\bar{S})\cup\bigcup_{Q\in S} D_Q$. If $T\in D_Q$ $(D_Q\neq D(Q))$, for some $Q\in S$, then $v_{M_{\nu_P-1}}(T)=1$ or 0. If $T\in D_Q$, where D_Q is $D(Q)$, for some $Q\in S$, then, since $D(Q)$ is $\{Q\}$, $T\in D'$. By ind. hyp. for D', $v_{M_{\nu_P-1}}(T)=1$ or 0. If $T\in D'\cap\bar{S}$, then, by ind. hyp. for D', $v_{M_{\nu_P-1}}(T)=1$ or 0. Hence the lemma holds.

<u>Lemma 7.</u>

Let $P(A)$ be a standard wff such that $v_{M_\lambda}(P(A))=1$ or 0. Consider any gen. dep. set D' of $P(A)$, such that, in the process of construction, (ii) is not applied to any atomic wff of the form $C\in A$. If, for all $Q(A)\in D'$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$, then $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

<u>Proof.</u> By induction on the stages of construction of gen. dep. sets of <u>all</u> standard wffs $P(A)$ such that $v_{M_\lambda}(P(A))=1$ or 0, such that (ii) is not applied to any atomic wff of the form $C\in A$.

(i) Let the gen. dep. set of $P(A)$ be $D(P(A))$. Then, by lemma 4, the lemma holds.

(ii) Let the gen. dep. set of $P(A)$ be $D(C_{P(A)})$, where $P(A)$ is an atomic wff. We need only consider $P(A)$ in the form $A\in\{X : Q\}$. Hence $C_{P(A)}$ is $Q(A)$. $v_{M_\lambda}(Q(A))=1$ or 0. By the lemma condition, if $R(A)\in D(C_{P(A)})$ then $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))$. Hence, by lemma 4, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. Therefore, $v_{M_\lambda}(B\in\{X : Q\})=v_{M_\lambda}(A\in\{X : Q\})$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iii) Let D' be a gen. dep. set of $P(A)$ and let $S\subseteq D'$. For each

$Q(A)\in S$, let $D_{Q(A)}$ be a gen. dep. set of $Q(A)$. Let the lemma hold for D' and the $D_{Q(A)}$'s. By the condition of the lemma, for all $T(A)\in(D'\cap\bar{S})\cup\bigcup_{Q(A)\in S} D_{Q(A)}$, $v_{M_\lambda}(T(B))=v_{M_\lambda}(T(A))$. Since $D_{Q(A)}\subseteq(D'\cap\bar{S})\cup\bigcup_{Q(A)\in S} D_{Q(A)}$, for all $Q(A)\in S$, by ind. hyp., $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$, for all $Q(A)\in S$. Also, for all $T(A)\in D'\cap\bar{S}$, $v_{M_\lambda}(T(B))=v_{M_\lambda}(T(A))$. Hence, if $U(A)\in D'$, $v_{M_\lambda}(U(B))=v_{M_\lambda}(U(A))$. By ind. hyp. for D', $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

Lemma 8.

If $v_{M_\lambda}(A\in C)=1$ or 0 then $A\in C$ has a gen. dep. set without any wffs of the form $A\in B$ for any B, except A. The gen. dep. sets so constructed are such that (ii) is not applied to any atomic wffs of the form $A'\in A$.

Proof. Let the wff $A\in C$ be W. The proof is by transfinite induction on ν_W. The ind. hyp. is that the lemma holds for all wffs $A\in D$ (call it X) such that $\nu_X<\nu_W$.

(i) $\nu_W=1$. Let $v_{M_1}(A\in C)=1$ or 0. Let A and C be different. Then $v_{M_0}(C_{A\in C})$ is 1 or 0. Hence $D(C_{A\in C})=\{1\}$ or $\{0\}$. This satisfies the lemma. If A is C, then $D(A\in C)=\{A\in C\}$ satisfies the lemma.

(ii) ν_W is a successor ordinal >1. Let $v_{M_{\nu_W}}(A\in C)=1$ or 0. If A is C, then $D(A\in C)=\{A\in C\}$ satisfies the lemma. If A and C are different, $v_{M_{\nu_W-1}}(Z(A))=1$ or 0, where $Z(A)$ is C_W. Hence, $D(Z(A))$ is a gen. dep. set of W. It has a subset S of all atomic wffs of the form $A\in B$, where B is not A. For all Q, if $Q\notin S$, then $v_{M_{\nu_W-1}}(Q)=1$ or 0. Hence, by ind. hyp., all these wffs $Q\notin S$ have gen. dep. sets D_Q without wffs of the above form. Form the set $(D(Z(A))\cap\bar{S})\cup\bigcup_{Q\notin S} D_Q$, which has no atomic wffs of the above form. This is a gen. dep. set of W which satisfies the lemma.

Lemma 9.

If $A\in C\leftrightarrow A\in D$ has the value 1 in M_λ, for all A, then $C\in C\leftrightarrow D\in D$ has the value 1 in M_λ.

Proof. Call $C\in C$, W. Let $v_{M_\lambda}(W)=1$ or 0. By lemma 8, W has a gen. dep. set D' without atomic wffs of certain forms and constructed in a certain way. For the sake of lemma 8 the right hand C of $C\in C$ is regarded as different from the left hand C. So (ii) is applied in forming a gen. dep. set of $C\in C$, but apart from this one instance all the usual conditions apply. By lemma 6, all members of D' have the value 1 or 0 in M_{ν_W-1}, since, by lemma 8, D' can be constructed so that it is not D(W). Hence W is not a member of D'. Hence D' has atomic wffs containing C, only of the form $A\in C$ or not at all. By the condition of the lemma, if $Q(C)\in$ D' then $v_{M_\lambda}(Q(D))=v_{M_\lambda}(Q(C))$. By lemma 7, $v_{M_\lambda}(D\in C)=v_{M_\lambda}(C\in C)$. Since (ii) was applied to $C\in C$ in forming the gen. dep. set D', the substitution of D for C occurs only in the left hand C of $C\in C$. By the condition of the lemma, $v_{M_\lambda}(D\in D)=v_{M_\lambda}(D\in C)$ and hence $v_{M_\lambda}(D\in D)=v_{M_\lambda}(C\in C)$. Similarly, by letting $D\in D$ be W and substituting C for D, $v_{M_\lambda}(C\in C)=v_{M_\lambda}(D\in D)$. Hence the lemma is proved.

Theorem 3.

The Axiom of Extensionality is valid in M_λ.

Proof. We will prove : If $V\in C \leftrightarrow V\in D$ is valid in M_λ, then $C\in Z \leftrightarrow D\in Z$ is valid in M_λ. Let $v_{M_\lambda}(C\in C')=1$ or 0. By lemma 8, $C\in C'$ has a gen. dep. set D' without any wffs of the form $C\in B$, for any B except C. Hence the only occurrences of C in D' are of the form $A\in C$ (A is not C) and $C\in C$. Because of the condition of the theorem and because of lemma 9, if $Q(C)\in D'$, then $v_{M_\lambda}(Q(D))=v_{M_\lambda}(Q(C))$. By lemma 7, $v_{M_\lambda}(D\in C')=v_{M_\lambda}(C\in C')$. Hence $C\in Z \leftrightarrow D\in Z$ is valid in M_λ and the theorem is shown.

Since the model M_λ with domain D was constructed using only set theory formalisable within Z-F, the Axioms of Abstraction and Extensionality are relatively consistent to Z-F.

CHAPTER 6.

A 3-VALUED CLASS THEORY AVOIDING THE PARADOXES AND CONTAINING NBG.

In this chapter, I wish to present the 3-valued class theory of the previous chapter but with NBG embedded in it. This is what was mentioned in the Introduction and in the previous chapter.

The formalisation is as follows :

Primitives.

1. u,v,w,x,y,z,.... (variables over special classes, i.e. the classes of NBG.)

2. U,V,W,X,Y,Z,..... (variables over classes.)

3. $\in$ (is a member of.)

4. $\sim$, $\rightarrow$, A (connectives and quantifier of the Lukasiewicz 3-valued logic.)

Formation Rules.

1. For variables x, y, X, Y, the following are atomic wffs : $x \in y$, $x \in X$, $X \in x$, $X \in Y$.

2. The propositional constants 1, 0, $\frac{1}{2}$ are atomic wffs.

3. If B and C are wffs and x and X are variables then $\sim B$, $B \rightarrow C$, (Ax)B, (AX)B are wffs.

Definitions.

$X = Y =_{df} (AZ)(Z \in X \leftrightarrow Z \in Y)$.

$V(X) =_{df} (AZ) \quad C(Z \in X)$.

The definitions of NBG, each one distinguished from any similar definition for classes in general by the symbol 's'. For example :

$x \overset{s}{=} y =_{df} (Az)(z \in x \equiv z \in y)$.

The set consisting of the single set x is represented as $\{x\}_s$. The power set of x is $\mathcal{P}_s(x)$. The definitions are taken from Mendelson, [17], p.159-188.

$(Ax')\phi(x') =_{df} (Ax)(M(x) \supset \phi(x))$.

$(Sx')\phi(x') =_{df} (Sx)(M(x) \,\&\, \phi(x))$.

$u', v', w', x', y', z', \ldots$ are variables over sets.

$SCl(X) =_{df} (Sx)(Az)(z \in x \leftrightarrow z \in X)$. (X is a special class in that it has the same special class members as some special class but X may not lie in the range of the special class variables.)

<u>Axioms.</u>

T. $x \overset{s}{=} y \supset . x \in z \equiv y \in z$.

P. $(Ax')(Ay')(Sz')(Au')(u' \in z' \equiv u' \overset{s}{=} x' \vee u' \overset{s}{=} y')$.

N. $(Sx')(Ay')\sim y' \in x'$.

B. $(Sz)(Ax_1')\ldots(Ax_n')(\langle x_1', \ldots, x_n' \rangle_s \in z \equiv \phi(x_1', \ldots, x_n', y_1, \ldots, y_m))$, where only set variables are quantified in ϕ.

U. $(Ax')(Sy')(Au')(u' \in y' \equiv (Sv')(u' \in v' \,\&\, v' \in x'))$.

W. $(Ax')(Sy')(Au')(u' \in y' \equiv u' \subseteq_s x')$.

S. $(Ax')(Ay)(Sz')(Au')(u' \in z' \equiv u' \in x' \,\&\, u' \in y)$.

R. $(Ax')(Un_s(x) \supset (Sy')(Au')(u' \in y' \equiv (Sv')(\langle v', u' \rangle_s \in x \,\&\, v' \in x')))$.

I. $(Sx')(0 \in x' \,\&\, (Au')(u' \in x' \supset u' \cup \{u'\}_s \in x'))$.

A. $(SY)(AX)(X \in Y \leftrightarrow \phi(X, z_1, \ldots, z_m, Z_1, \ldots, Z_n))$, where ϕ is either a propositional constant or constructed from atomic wffs of forms, $U \in V$, $U \in v$, $u \in V$, $u \in v$, by using only $\sim$, &, A.

E. $X=Y \supset (AZ)(X \in Z \leftrightarrow Y \in Z)$.

General Axioms.

1. $(AX)\phi(X) \rightarrow (Ax)\phi(x)$.

2. $(Az)(z \in x \leftrightarrow z \in X) \supset (Aw)(x \in w \leftrightarrow X \in w)$.

3. $FSCl(X) \supset F(X \in x)$.

4. $PSCl(X) \supset P(X \in x)$.

5. $C(x \in y)$.

The Axioms T to I are the axioms of NBG as in Mendelson. The Axioms of Choice, Restriction, Constructibility and the Generalised Continuum Hypotnesis can be added if one wishes. As will be shown later, one can prove the relative consistency of the above system with or without these additional axioms. The Axioms A and E are the 3-valued Axioms of Abstraction and Extensionality, resp. General Axiom 1 follows from the fact that every special class is a class. General Axioms 2,3 and 4 determine the values for class membership of special classes. This determination is the one used in constructing the model of the theory. There is a degree of arbitrariness about it, especially for the values arising from General Axiom 4. However, it fits in with the model and I cannot think of anything else that will. Also certain interesting consequences follow from it, e.g. $x \overset{S}{=} y \supset x=y$. General Axiom 5 asserts the two-valuedness of membership between special classes.

The theorems following from the above axioms will be the theorems of NBG for special classes, the theorems of the previous chapter for classes in general, and theorems following from the General Axioms that relate special classes with classes in general. Hence the only theorems left to deal with are the ones following from the General Axioms as the others can be found in Mendelson or in Chapter 5.

T.1. $\underline{(Sx)\emptyset(x) \rightarrow (SX)\emptyset(X).}$

Gen. Ax. 1 : $(AX)\sim\emptyset(X) \rightarrow (Ax)\sim\emptyset(x)$ ______(1)

(1) : $\sim(Ax)\sim\emptyset(x) \rightarrow \sim(AX)\sim\emptyset(X)$ ______(2)

(2) : T.1.

T.2. $\underline{x \overset{s}{=} y \supset x = y.}$

Hyp : $x \overset{s}{=} y$ ______(1)

(1), Defn. $\overset{s}{=}$: $(Aw)(w \in x \equiv w \in y)$ ______(2)

(1), Ax.T : $(Az)(x \in z \equiv y \in z)$ ______(3)

Hyp : $TSCl(Z)$ ______(4)

(4), Defn.SCl : $(Aw)(w \in Z \leftrightarrow w \in z_1)$ ______(5) (z_1 is a constant.)

(5), Gen.Ax.2 : $(Au)(Z \in u \leftrightarrow z_1 \in u)$ ______(6)

(6) : $Z \in x \leftrightarrow z_1 \in x$ ______(7)

(6) : $Z \in y \leftrightarrow z_1 \in y$ ______(8)

Gen.Ax.5, (2) : $z_1 \in x \leftrightarrow z_1 \in y$ ______(9)

(7), (8), (9) : $Z \in x \leftrightarrow Z \in y$ ______(10)

(4), (10) : $TSCl(Z) \supset . Z \in x \leftrightarrow Z \in y$ ______(11)

Hyp : $FSCl(Z)$ ______(12)

(12), Gen.Ax.3 : $F(Z \in x)$ & $F(Z \in y)$ ______(13)

(13) : $Z \in x \leftrightarrow Z \in y$ ______(14)

(12), (14) : $FSCl(Z) \supset . Z\in x \leftrightarrow Z\in y$ ____(15)

Hyp : $PSCl(Z)$ ____(16)

(16), Gen.Ax.4 : $P(Z\in x)$ & $P(Z\in y)$ ____(17)

(17) : $Z\in x \leftrightarrow Z\in y$ ____(18)

(16), (18) : $PSCl(Z) \supset . Z\in x \leftrightarrow Z\in y$ ____(19)

(11), (15), (19) : $Z\in x \leftrightarrow Z\in y$ ____(20)

(20), Defn.= : $x=y$ ____(21)

(1), (21) : T.2.

T.3. <u>$x \overset{s}{=} y \equiv x=y$.</u>

Gen.Ax.1 : $(AZ)(Z\in x \leftrightarrow Z\in y) \supset (Az)(z\in x \leftrightarrow z\in y)$ ____(1)

Gen.Ax.5, (1), Defns.$\overset{s}{=}$,= : $x=y \supset x \overset{s}{=} y$ ____(2)

(2), T.2 : T.3.

T.4. <u>$(Aw)(x\in w \leftrightarrow X\in w) \supset (Az)(z\in x \leftrightarrow z\in X)$.</u>

Hyp : $(Aw)(x\in w \leftrightarrow X\in w)$ ____(1)

(1) : $x\in\{x\}_s \leftrightarrow X\in\{x\}_s$ ____(2)

(2) : $T(X\in\{x\}_s)$ ____(3)

(3), Gen.Axs.3 and 4 : $TSCl(X)$ ____(4)

(4), Defn.SCl : $(Az)(z\in y_1 \leftrightarrow z\in X)$ ____(5) (y_1 is a constant.)

(5), Gen.Ax.2 : $(Aw)(y_1\in w \leftrightarrow X\in w)$ ____(6)

(3), (6) : $T(y_1\in\{x\}_s)$ ____(7)

(7), Defn.$\{x\}_s$: $y_1 \overset{s}{=} x$ ____(8)

(8), T.3 : $y_1=x$ ____(9)

(5), (9) : $(Az)(z\in x \leftrightarrow z\in X)$ ____(10)

(1), (10) : T.4.

T.5. $\underline{(Az)(z \in x \leftrightarrow z \in X) \equiv (Aw)(x \in w \leftrightarrow X \in w).}$

Gen.Ax.2, T.4 : T.5.

T.6. $\underline{PSCl(X) \equiv P(X \in x).}$

Hyp : $P(X \in x)$ ______(1)

(1), Gen.Ax.3 : $\sim FSCl(X)$ ______(2)

Hyp : $TSCl(X)$ ______(3)

(3), Defn.SCl : $(Az)(z \in y_1 \leftrightarrow z \in X)$ ______(4) (y_1 is a constant.)

(4), Gen.Ax.2 : $(Aw)(y_1 \in w \leftrightarrow X \in w)$ ______(5)

(5), Gen.Ax.5 : $C(X \in x)$ ______(6)

(1), (3), (6) : $\sim TSCl(X)$ ______(7)

(2), (7) : $PSCl(X)$ ______(8)

(1), (8) : $P(X \in x) \supset PSCl(X)$ ______(9)

(9), Gen.Ax.4 : T.6.

T.7. $\underline{x \overset{s}{=} y \leftrightarrow x = y.}$

Hyp : $P(x=y)$ ______(1)

(1), Defn.= : $P(AZ)(Z \in x \leftrightarrow Z \in y)$ ______(2)

(2) : $P(Z_1 \in x \leftrightarrow Z_1 \in y)$ ______(3) (Z_1 is a constant.)

(3) : $P(Z_1 \in x)$ v $P(Z_1 \in y)$ ______(4)

(4), T.6 : $PSCl(Z_1)$ ______(5)

(5), Gen.Ax.4 : $P(Z_1 \in x)$ & $P(Z_1 \in y)$ ______(6)

(6) : $T(Z_1 \in x \leftrightarrow Z_1 \in y)$ ______(7)

(1), (3), (7) : $C(x=y)$ ______(8)

(8), T.3 : T.7.

T.8. <u>$TSCl(x)$.</u>

Defn.SCl : $SCl(X) \leftrightarrow (Sy)(Az)(z\in y \leftrightarrow z\in X)$ ______(1)

(1), Gen.Ax.1 : $SCl(x) \leftrightarrow (Sy)(Az)(z\in y \leftrightarrow z\in x)$ ______(2)

(2) : T.8.

T.9. <u>$PSCl(H)$.</u>

Defn.SCl : $SCl(H) \leftrightarrow (Sx)(Az)(z\in x \leftrightarrow z\in H)$ ______(1)

Defn.H : $\sim F(z\in x \leftrightarrow z\in H)$ ______(2)

(2), Gen.Ax.5 : $P(z\in x \leftrightarrow z\in H)$ ______(3)

(3) : $P(Sx)(Az)(z\in x \leftrightarrow z\in H)$ ______(4)

(1), (4) : T.9.

T.10. <u>$P(H\in x)$.</u>

T.9, Gen.Ax.4 : T.10.

T.11. <u>$\sim V(x)$.</u>

T.10, Defn.V : T.11.

We now introduce the notion of absoluteness for functions and wi.s. f is absolute if $f_s(x_1,\dots,x_n)=f(x_1,\dots,x_n)$, for all $x_1,\dots,x_n$, where f_s is defined the same way as f except that all quantification is with special class variables. ϕ is absolute if $\phi_s(x_1,\dots,x_n) \leftrightarrow \phi(x_1,\dots,x_n)$, for all $x_1,\dots,x_n$, where ϕ_s is defined the same way as ϕ except that all quantification is with special class variables.

By T.7, = is absolute, since $x \overset{s}{=} y \leftrightarrow x=y$. The union, $\cup$, is absolute since $z\in x\cup y \leftrightarrow z\in x \vee z\in y$ and $z\in x\cup_s y \leftrightarrow z\in x \vee z\in y$. The intersection, $\cap$, is absolute since $z\in x\cap y \leftrightarrow z\in x \,\&\, z\in y$ and $z\in x\cap_s y \leftrightarrow z\in x \,\&\, z\in y$.

[Note that in the Axiom I, it does not matter whether it is written with $u' \cup_s \{u'\}_s$ or with $u' \cup \{u'\}_s$.]

The complement, $\bar{\ }$, is absolute, since $z \in \bar{x} \leftrightarrow \sim(z \in x)$ and $z \in \bar{x}^s \leftrightarrow \sim(z \in x)$. The null class, $\emptyset$, is absolute, since $z \in \emptyset \leftrightarrow 0$ and $z \in 0 \leftrightarrow 0$. The universal class, U, is absolute, since $z \in U \leftrightarrow 1$ and $z \in V \leftrightarrow 1$. [V is the special class of all sets because proper (special) classes do not belong to special classes.]

However, $\{\ \}$ is not absolute because $z \in \{x\} \leftrightarrow (AW)(\sim W \in z \text{ v } W \in x . \& . \sim W \in x \text{ v } W \in z)$ and $z \in \{x\}_s \leftrightarrow (Aw)(\sim w \in z \text{ v } w \in x . \& . \sim w \in x \text{ v } w \in z)$. By T.10, $\sim T(AW)(\sim W \in z \text{ v } W \in x . \& . \sim W \in x \text{ v } W \in z)$ and hence $\sim T(z \in \{x\})$. But $T(x \in \{x\}_s)$ and hence $\sim T(x \in \{x\}_s \leftrightarrow x \in \{x\})$. $\mathcal{P}$ is also not absolute because $z \in \mathcal{P}(x) \leftrightarrow (AW)(\sim W \in z \text{ v } W \in x)$ and $z \in \mathcal{P}_s(x) \leftrightarrow (Aw)(\sim w \in z \text{ v } w \in x)$. By T.10, $\sim T(z \in \mathcal{P}(x))$. But $T(x \in \mathcal{P}(x))$ and hence $\sim T(x \in \mathcal{P}(x) \leftrightarrow x \in \mathcal{P}_s(x))$.

The connectives needed to avoid the non-absoluteness of $\{\ \}$ and $\mathcal{P}$ are not available for substitution into the Abstraction Axiom. The strange properties of $\{\ \}$ and $\mathcal{P}$ in the last chapter show up even when the domain is restricted to special classes. Note also that the Boolean operations which are well-behaved in the last chapter continue to be so when the domain is restricted to special classes.

The next task is to prove the consistency, relative to Z-F, of the above theory. The method is similar to that used in the last chapter, but differs from it in that M_0 contains a model of NBG and the method of generating the sequence of structures, $M_0 \leq M_1$

$\leq \ldots . \leq M_{\mu} \leq \ldots .$, is more complicated.

Take any model N of NBG whose domain is a denumerable set. The domain will consist of special class constants and the membership between any two of these constants will be determined as true or false in N. To construct the model of the whole system, we need to extend the above wffs by adding the special class constants of the above model of NBG, a,b,c,...., and some terms to be defined. The domain of the model will consist of some of these terms as well as the special class constants. We give the formation rules for terms and wffs as follows :

1. If x and y are special class variables, a and b are special class constants, and X and Y are class variables, then $a\in b$, $a\in x$, $x\in a$, $a\in X$, $X\in a$, $x\in y$, $x\in X$, $X\in x$, $X\in Y$, are atomic wffs.

2. Any combination of wffs using $\sim, \rightarrow$, A as in the Lukasiewicz 3-valued logic is a wff.

3. A propositional constant (i.e. 1, $\frac{1}{2}$ or 0) is an atomic wff.

4. A propositional constant or a wff constructed from atomic wffs using only $\sim$, &, A is a standard wff.

5. If P is a standard wff and X is a class variable, then $\{X : P\}$ is a term.

6. If $\{X : P\}$ and $\{X : Q\}$ are terms, y is a special class variable, a is a special class constant and Y is a class variable, then $\{X : P\}\in a$, $a\in\{X : P\}$, $\{X : P\}\in y$, $y\in\{X : P\}$, $\{X : P\}\in Y$, $Y\in\{X : P\}$, $\{X : P\}\in\{X : Q\}$ are all atomic wffs.

We construct a model for the axioms with domain the set D of all special class constants and all constant terms $\{X : P\}$, i.e. P is a standard wff and either has no free variables at all or has X as its only free variable. Let D^S denote the set of all special class constants and so $D-D^S$ is the set of all constant terms. We shall use constants A, B, C, ..., for members of D. Non-constant terms can be defined from these as follows: Associate with any term $\{X : P(X, z_1, \ldots, z_m, Z_1, \ldots, Z_n)\}$, for which $z_1, \ldots, z_m, Z_1, \ldots, Z_n$ are the only free variables, the function which for constants $a_1, \ldots, a_m$ of D^S and $A_1, \ldots, A_n$ of D takes as value the constant term $\{X : P(X, a_1, \ldots, a_m, A_1, \ldots, A_n)\}$ of D.

Let any specification of values including the value assignments already given to members of D^S in the model N, for all the constant atomic wffs $A \in B$, where A and B are members of D, be called a st ture on D. Let $v_M(P)$ denote the value of the constant wff P given by the structure M on D. Also let $v_M(1)=1$, $v_M(0)=0$ and $v_M(\frac{1}{2})=\frac{1}{2}$ Define $M_1 \leq M_2$ for two structures M_1 and M_2 on D as, for any constant atomic wff P if $v_{M_1}(P)=1$ then $v_{M_2}(P)=1$ and if $v_{M_1}(P)=0$ then $v_{M_2}(P)=0$. '$\leq$' defines a partial ordering on the set of structures since (i) $M \leq M$, (ii) if $M_1 \leq M_2$ and $M_2 \leq M_3$ then $M_1 \leq M_3$ and (iii) if $M_1 \leq M_2$ and $M_2 \leq M_1$ then $M_1 = M_2$ (i.e. M_1 and M_2 are the same structure).

From now on, when mentioning values of wffs in a structure it is automatically assumed that the wffs are constant ones, i.e. they have no free variables.

Lemma 1.

Let M and M' be two structures on D, such that $M \leq M'$. Then, for any standard wff P, if $v_M(P)=1$ then $v_{M'}(P)=1$ and if $v_M(P)=0$ then $v_{M'}(P)=0$.

Proof. By induction on wff evaluation procedure. This means that we start at the values of all the constant atomic wffs obtained by substitution for free variables in P, and then build up the value of P from these values according to the connectives and quantifiers in the Lukasiewicz logic. If P is an atomic wff, the lemma holds.

(i) Let $v_M(\sim Q)=1$. Then $v_M(Q)=0$. By ind. hyp., $v_{M'}(Q)=0$. Hence $v_{M'}(\sim Q)=1$.

Let $v_M(\sim Q)=0$. Then $v_M(Q)=1$. By ind. hyp., $v_{M'}(Q)=1$. Hence $v_{M'}(\sim Q)=0$.

(ii) Let $v_M(Q \,\&\, R)=1$. Then $v_M(Q)=1$ and $v_M(R)=1$. By ind. hyp., $v_{M'}(Q)=1$ and $v_{M'}(R)=1$. Hence $v_{M'}(Q \,\&\, R)=1$.

Let $v_M(Q \,\&\, R)=0$. Then $v_M(Q)=0$ or $v_M(R)=0$. By ind. hyp., $v_{M'}(Q)=0$ or $v_{M'}(R)=0$. Hence $v_{M'}(Q \,\&\, R)=0$.

(iii) Let $v_M((Ax)Q(x))=1$. Then $v_M(Q(x))=1$ for all $x \in D^S$. By ind. hyp., $v_{M'}(Q(x))=1$ for all $x \in D^S$. Hence $v_{M'}((Ax)Q(x))=1$.

Let $v_M((Ax)Q(x))=0$. Then $v_M(Q(x))=0$ for some $x \in D^S$. By ind. hyp., $v_{M'}(Q(x))=0$ for some $x \in D^S$. Hence $v_{M'}(Ax)Q(x))=0$.

(iv) The case for (AX)Q(X) is similar to (iii).

Define the structure M_0 as follows:

If $A \notin D^S$ or $B \notin D^S$, then $v_{M_0}(A \in B) = \frac{1}{2}$.

If $A \in D^S$ and $B \in D^S$, then $v_{M_0}(A \in B) = 1$ if $A \in B$ is true in the model N and $= 0$ if $A \in B$ is false in the model N.

Hence M_0 with domain D^S is a model of NBG satisfying all the axioms.

The model of the whole system will be the limit of a sequence of structures. $M_0 \leq M_1 \leq \ldots \leq M_\mu \ldots$, on D. Assuming M_μ defined for some ordinal , $M_{\mu+1}$ is defined as follows:

For all standard wffs P, $v_{M_{\mu+1}}(A \in \{X : P(X)\}) = v_{M_\mu}(P(A))$.

If $\sim z \in A \vee z \in a . \& . \sim z \in a \vee z \in A$ is valid in M_μ for some a, then, for all b, $v_{M_{\mu+1}}(A \in b) = v_{M_0}(a \in b)$.

If $(Ax)(Sz)(z \in A \ \& \sim z \in x . v . z \in x \ \& \sim z \in A)$ has the value 1 in M_μ then, for all b, $v_{M_{\mu+1}}(A \in b) = 0$.

If neither $(Sx)(Az)(\sim z \in A \vee z \in x . \& . \sim z \in x \vee z \in A)$ nor $(Ax)(Sz)(z \in A \ \& \sim z \in x . v . z \in x \ \& \sim z \in A)$ have the value 1 in M_μ then, for all b, $v_{M_{\mu+1}}(A \in b) = \frac{1}{2}$.

For a limit ordinal μ, on the assumption that $M_\nu \leq M_\tau$ for all $\nu \leq \tau$, for all $\tau < \mu$, for all atomic wffs P, if $v_{M_\nu}(P) = 1$ for some $\nu < \mu$ then $v_{M_\mu}(P) = 1$, if $v_{M_\nu}(P) = 0$ for some $\nu < \mu$ then $v_{M_\mu}(P) = 0$, and if $v_{M_\nu}(P) = \frac{1}{2}$ for all $\nu < \mu$ then $v_{M_\mu}(P) = \frac{1}{2}$.

<u>Lemma 2.</u>

$M_\nu \leq M_\mu$, for all $\nu \leq \mu$

<u>Proof.</u> By transfinite induction on μ. The induction hypothesis is : $M_\nu \leq M_\tau$ for all $\nu \leq \tau$, for all $\tau < \mu$.

(i) $\mu=0.$ $M_0 \leq M_0$.

(ii) μ is a successor ordinal.

(A) Let $v_{M_\nu}(A\in\{X : P\})=1$. There is a $\eta<\nu$ such that $v_{M_\eta}(P(A))=1$ by the method of construction of the structures. Since $\eta\leq\mu-1$, $M_\eta\leq M_{\mu-1}$ by ind. hyp. Hence $v_{M_{\mu-1}}(P(A))=1$. By the construction of M_μ, $v_{M_\mu}(A\in\{X : P\})=1$. Similarly, if $v_{M_\nu}(A\in\{X : P\})=0$ then $v_{M_\mu}(A\in\{X : P\})=0$.

(B) Let $v_{M_\nu}(A\in b)=1$ (or 0) There is a $\eta<\nu$ such that $v_{M_\eta}((Sx)(Az)(\sim z\in A \vee z\in x . \& . \sim z\in x \vee z\in A))=1$ or $v_{M_\eta}((Ax)(Sz)(z\in A \;\&\; \sim z\in x . \vee . z\in x \;\&\; \sim z\in A))=1$.

(a) Let $v_{M_\eta}((Sx)(Az)(\sim z\in A \vee z\in x . \& . \sim z\in x \vee z\in A))=1$. Then $v_{M_{\eta+1}}(A\in b)=v_{M_0}(a\in b)=1$ (or o), for some a. Since $\eta\leq\mu-1$, $M_\eta\leq M_{\mu-1}$ by the ind. hyp. Hence $v_{M_{\mu-1}}((Sx)(Az)(\sim z\in A \vee z\in x . \& . \sim z\in x \vee z\in A))=1$ and $v_{M_\mu}(A\in b)=v_{M_0}(a\in b)=1$ (or 0), for some a.

(b) Let $v_{M_\eta}((Ax)(Sz)(z\in A \;\&\; \sim z\in x . \vee . z\in x \;\&\; \sim z\in A))=1$. If $v_{M_\nu}(A\in b)=1$, this does not apply. Let $v_{M_\nu}(A\in b)=0$. Since $\eta\leq\mu-1$, $M_\eta\leq M_{\mu-1}$, by the ind. hyp. Hence $v_{M_{\mu-1}}((Ax)(Sz)(z\in A \;\&\; \sim z\in x . \vee . z\in x \;\&\; \sim z\in A))=1$ and $v_{M_\mu}(A\in b)=0$.

(iii) μ is a limit ordinal.

Let $\nu<\mu$. Let $v_{M_\nu}(A\in B)=1$. Then $v_{M_\mu}(A\in B)=1$ by definition of M_μ. Similarly when $v_{M_\nu}(A\in B)=0$ then $v_{M_\mu}(A\in B)=0$. If $\nu=\mu$, $M_\nu\leq M_\mu$.

Lemma 3

There is an ordinal λ of the second number class such that $M_\lambda=M_{\lambda+1}$

Proof. The increasing chain of structures, $M_0\leq M_1\leq\ldots\leq M_\mu\leq\ldots$, can be regarded as two increasing chains of subsets of the denumerable set of all atomic wffs of the form $A\in B$ One chain is of those atomic wffs taking the value 1 and the other is of those taking the value 0 If $M_\nu = M_{\nu+1}$ then $M_\nu = M_\mu$ for all ordinals μ such that $\nu \leq \mu$, since, by the method of construction of the structures, there is no way of changing the values of any atomic wffs. There is a denumerable set of ordinals μ such that $M_\mu \neq M_{\mu+1}$. But the set of all ordinals of the second number class is non-denumerable, and hence for some λ in this class, $M_\lambda = M_{\lambda+1}$.

Now it is required to show that M_λ is the required model.

Theorem 1.

All the axioms of NBG are valid in M_λ.

Proof. By the definitions of M_0 and the domain D^S, M_0 with D^S as domain is a model of NBG. By lemma 2, if $v_{M_0}(A\in B)=1$ (or 0) then $v_{M_\lambda}(A\in B)=1$ (or 0). Hence M_λ with domain D^S is a model of NBG.

Theorem 2.

$Y\in\{X : P\} \leftrightarrow P(Y)$ is valid in M_λ.

Proof Let $v_{M_\lambda}(A\in\{X : P\})=1$. Let ν be the least ordinal such that $v_{M_\nu}(A\in\{X : P\})=1$. ν is a successor ordinal. Hence $v_{M_{\nu-1}}(P(A))$ $=1$. Since $\nu-1\leq\lambda$, $M_{\nu-1}\leq M_\lambda$, by lemma 2. Since P is a standard wff, by lemma 1, $v_{M_\lambda}(P(A))=1$. Similarly, if $v_{M_\lambda}(A\in\{X : P\})=0$, then $v_{M_\lambda}(P(A))=0$. Let $v_{M_\lambda}(P(A))=1$. Then $v_{M_{\lambda+1}}(A\in\{X : P\})=1$.

Since $M_\lambda = M_{\lambda+1}$, $v_{M_\lambda}(A \in \{X : P\}) = 1$. Similarly, if $v_{M_\lambda}(P(A)) = 0$, then $v_{M_\lambda}(A \in \{X : P\}) = 0$.

Theorem 3.

The Abstraction Axiom (A) is valid in M_λ.

Proof. By theorem 2, for any standard wff P, $Y \in \{X : P\} \leftrightarrow P(Y)$ is valid in M_λ. Therefore $(SZ)(AX)(X \in Z \leftrightarrow P(X, y_1, \ldots, y_m, Y_1, \ldots, Y_n))$ is valid in M_λ, for all wffs P which are either propositional constants or constructed from atomic wffs of forms, $U \in V$, $U \in v$, $u \in V$, $u \in v$ by using ~, &, A only; since all wffs of this sort are standard wffs.

Let P be a standard wff such that $v_{M_\lambda}(P) = 1$ or 0. Let ν_P be the least ordinal such that $v_{M_{\nu_P}}(P) = 1$ or 0. Form the set of all constant atomic wffs P (i.e. atomic wffs of P with all substitutions made for any variables that occur in them) which take the value 1 or 0 in M_{ν_P}. Call this the dependent set of P, D(P).

Lemma 4.

Let P(A) be a standard wff such that $v_{M_\lambda}(P(A)) = 1$ or 0. If, for each $Q(A) \in D(P(A))$ $v_{M_\lambda}(Q(B)) = v_{M_\lambda}(Q(A))$, then $v_{M_\lambda}(P(B)) = v_{M_\lambda}(P(A))$.

Proof. By induction on wff evaluation procedure. Let P(A) be an atomic wff such that $v_{M_\lambda}(P(A)) = 1$ or 0. Then $D(P(A)) = \{P(A)\}$. Hence $v_{M_\lambda}(P(B)) = v_{M_\lambda}(P(A))$.

(i) Let P(A) be ~R(A) Since $D(\sim R(A)) = D(R(A))$, for each $Q(A) \in D(R(A))$, $v_{M_\lambda}(Q(B)) = v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B)) = v_{M_\lambda}(R(A))$.

Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(ii) Let $P(A)$ be $R(A)$ & $S(A)$ and $v_{M_\lambda}(R(A)\ \&\ S(A))=1$. Then $v_{M_\lambda}(R(A))=1$ and $v_{M_\lambda}(S(A))=1$. Since $\nu_{R(A)} \leq \nu_{R(A)\ \&\ S(A)}$; $D(R(A)) \subseteq D(R(A)\ \&\ S(A))$. Hence, for each $Q(A) \in D(R(A))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))$. Similarly, $v_{M_\lambda}(S(B))=v_{M_\lambda}(S(A))$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iii) Let $P(A)$ be $R(A)$ & $S(A)$ and $v_{M_\lambda}(R(A)\ \&\ S(A))=0$. Then $v_{M_\lambda}(R(A))=0$ or $v_{M_\lambda}(S(A))=0$. Since $\nu_{R(A)}=\nu_{R(A)\ \&\ S(A)}$ or $\nu_{S(A)}=\nu_{R(A)\ \&\ S(A)}$, $D(R(A)) \subseteq D(R(A)\ \&\ S(A))$ or $D(S(A)) \subseteq D(R(A)\ \&\ S(A))$. Hence, as above, $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))=0$ or $v_{M_\lambda}(S(B))=v_{M_\lambda}(S(A))=0$. Hence $v_{M_\lambda}(R(B)\ \&\ S(B))=0$ and $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iv) Let $P(A)$ be $(AZ)R(A,Z)$ and $v_{M_\lambda}((AZ)R(A,Z))=1$. Then $v_{M_\lambda}(R(A,Z))=1$ for all Z. Since $\nu_{R(A,Z)} \leq \nu_{(AZ)R(A,Z)}$ for all Z, then $D(R(A,Z)) \subseteq D((AZ)R(A,Z))$ for all Z. Hence for each $Q(A) \in D(R(A,Z))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B,Z))=v_{M_\lambda}(R(A,Z))$. Since this holds for all Z, $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(v) Let $P(A)$ be $(AZ)R(A,Z)$ and $v_{M_\lambda}((AZ)R(A,Z))=0$. Then $v_{M_\lambda}(R(A,Z))=0$ for some Z. Since $\nu_{R(A,Z)}=\nu_{(AZ)R(A,Z)}$ for some Z, say C, then $D(R(A,C)) \subseteq D((AZ)R(A,Z))$. For each $Q(A) \in D(R(A,C))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B,C))=v_{M_\lambda}(R(A,C))=0$. Hence $v_{M_\lambda}((AZ)R(B,Z))=0$ and $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(vi) Let $P(A)$ be $(Az)R(A,z)$ and $v_{M_\lambda}((Az)R(A,z))=1$. Similarly to (iv), $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(vii) Let $P(A)$ be $(Az)R(A,z)$ and $v_{M_\lambda}((Az)R(A,z))=0$. Similarly to (v), $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

Let P be an atomic wff (not 1 or 0) of the form $A \in \{X : Q(X)\}$ such that $v_{M_\lambda}(P)=1$ or 0. Define the corresponding standard wff of P, C_P, as Q(A). Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. Let P have dependent set, D(P). We define a general dependent set of P as follows:

(i) The dependent set D(P) of P is a gen. dep. set of P.

(ii) If $v_{M_\lambda}(R)=1$ or 0, R is an atomic wff (not 1 or 0) and C_R is defined for R, then $D(C_R)$ is a gen. dep. set of R.

(iii) Let D' be a gen. dep. set of P. Let $S \subseteq D'$. If $Q \in S$, let D_Q be a gen. dep. set of Q. Then $(D' \cap \bar{S}) \cup \bigcup_{Q \in S} D_Q$ is a gen. dep. set of P.

This assumes $v_{M_\lambda}(Q)=1$ or 0, for all $Q \in S$. Note that lemma 5 should be coupled with the definition of a gen. dep. set so that the assumption can be made before the construction of the gen. dep. sets D_Q.

Lemma 5.

Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. If D' is a gen. dep. set of P then, for each $Q \in D'$, $v_{M_\lambda}(Q)=1$ or 0.

Proof. The proof is the same as that, in (ii), C_R must be defined for R.

Lemma 6.

Let P be an atomic wff such that $v_{M_\lambda}(P)=1$ or 0 and such that C_P is defined. If D' is a gen. dep. set of P which is not D(P) then, for each $Q \in D'$, $v_{M_{\mu_P - 1}}(Q)=1$ or 0.

Proof. The proof is the same as that for lemma 6 of the last chapter, except that:

(I) The first ordinal that can be considered in the induction is 1. If $\nu_P=1$, then the only members Q of gen. dep. sets of P belong to $D(C_P)$ and satisfy $v_{M_0}(Q)=1$ or 0.

(II) In (ii) of the construction of gen. dep. sets of P, R must be such that C_R is defined.

Lemma 7.

Let P(A) be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. Consider any gen. dep. set D' of P(A), such that, in the process of construction, (ii) is not applied to any atomic wff of the form $C\in A$. If, for all $Q(A)\in D'$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$, then $v_{M_\lambda}(P(B))= v_{M_\lambda}(P(A))$.

Proof. The proof is the same as that of lemma 7 of the last chapter, except that in (ii), P(A) must be of the form $A\in\{X : Q(X)\}$ because $C_{P(A)}$ is defined.

Lemma 8.

If $v_{M_\lambda}(A\in C)=1$ or 0 then $A\in C$ has a gen. dep. set without any wffs of the form $A\in B$ for any B, except for A and for $B\in D^S$. The gen. dep. sets so constructed are such that (ii) is not applied to any atomic wffs of the form $A'\in A$.

Proof. Let the wff $A\in C$ be W. The proof is by the transfinite induction on ν_W, which is 0 or a successor ordinal. The ind. hyp. is that the lemma holds for all wffs $A\in D$ (call it X) such that $\nu_X < \nu_W$.

(i) $\underline{\nu_W = 0}$. $A \in D^S$ and $C \in D^S$. Let the gen. dep. set be $D(W)$, i.e. $\{W\}$. C_W is not defined.

(ii) $\underline{\nu_W \text{ is a successor ordinal.}}$

If $C \notin D^S$, let the gen. dep. set be $D(W)$. If C is A, then also let the gen. dep. set be $D(W)$. Otherwise, $v_{M_{\nu_W - 1}}(Z(A))=1$ or 0, where $Z(A)$ is C_W. Hence $D(Z(A))$ is a gen. dep. set of W. It has a subset S of all atomic wffs of the form $A \in B$, except where B is A or where $B \notin D^S$. For all Q, if $Q \in S$, then $v_{M_{\nu_W - 1}}(Q)=1$ or 0. Hence, by ind. hyp., all these wffs $Q \in S$ have gen. dep. sets D_Q without wffs of the above form. Form the set $(D(Z(A)) \cap \bar{S}) \cup \bigcup_{Q \in S} D_Q$, which has no atomic wffs of the above form. This is a gen. dep. set of W which satisfies the lemma.

Lemma 9.

If $Y \in A \leftrightarrow Y \in B$ is valid in M_λ, then $A \in A \leftrightarrow B \in B$ has value 1 in M_λ.

Proof. Call $A \in A$, W. Let $v_{M_\lambda}(W)=1$ or 0. By lemma 8, W has a gen. dep. set D' without atomic wffs of certain forms and constructed in a certain way For the sake of lemma 8, the right hand A of $A \in A$ is regarded as different from the left hand A. So (ii) is applied in forming a gen. dep. set of $A \in A$, but apart from this one instance all the usual conditions apply.

ν_W is either 0 or a successor ordinal.

(i) $\underline{\nu_W = 0}$. Then $A \in D^S$.

(A) Let $B \in D^S$. Then $z \in A \leftrightarrow Z \in B$ is valid in M_λ and hence in M_0. Because the Extensionality Axiom holds in NBG, $A \in y \leftrightarrow B \in y$ is valid in M_0. Hence $A \in A \leftrightarrow B \in A$ and, since $B \in A \leftrightarrow B \in B$, $A \in A \leftrightarrow B \in B$ is valid in M_0 and hence in M_λ.

(B) Let $B \notin D^S$. Then $z \in A \leftrightarrow z \in B$ is valid in M_λ. Hence by Lukasiewicz logic, $\sim z \in A \text{ v } z \in B \,\&\, z \in A \text{ v } \sim z \in B$ is valid in M_λ. Hence, by construction of $M_{\lambda+1}$, $v_{M_{\lambda+1}}(B \in A) = v_{M_0}(A \in A)$ and then, since $v_{M_\lambda}(B \in B) = v_{M_\lambda}(B \in A)$, $v_{M_\lambda}(B \in B) = v_{M_\lambda}(A \in A)$.

(ii) $\underline{\nu_W \text{ is a successor ordinal.}}$ Then $A \notin D^S$. By lemma 6, all members of D have the value of 1 or 0 in $M_{\nu_W - 1}$. Hence W is not a member of D'. Hence D' has atomic wffs containing A, only of the forms $A' \in A (A' \neq A)$ and $A \in B'$, where $B' \in D^S$.

Consider the atomic wff $A \in B'$.

(A) Let $\sim z \in A \text{ v } z \in a . \& . \sim z \in a \text{ v } z \in A$ be valid in M_λ for some a. Hence $v_{M_{\lambda+1}}(A \in B') = v_{M_0}(a \in B')$. By the condition of the lemma, $\sim z \in B \text{ v } z \in a . \& . \sim z \in a \text{ v } z \in B$ is valid in M_λ. Hence $v_{M_{\lambda+1}}(B \in B') = v_{M_0}(a \in B')$ and $v_{M_\lambda}(B \in B') = v_{M_\lambda}(A \in B')$.

(B) Let $(Ax)(Sz)(z \in A \,\&\, \sim z \in x . \text{v} . z \in x \,\&\, \sim z \in A)$ have the value 1 in M_λ. Hence $v_{M_{\lambda+1}}(A \in B') = 0$. [That is, the case (B) is not a possibility if $v_{M_\lambda}(A \in B') = 1$.] By the lemma condition, $(Ax)(Sz)(z \in B \,\&\, \sim z \in x . \text{v} . z \in x \,\&\, \sim z \in B)$ has the value 1 in M_λ. Hence $v_{M_{\lambda+1}}(B \in B') = 0$ and $v_{M_\lambda}(B \in B') = v_{M_\lambda}(A \in B')$. Hence, if $Q(A) \in D'$, $v_{M_\lambda}(Q(B)) = v_{M_\lambda}(Q(A))$. By lemma 7, $v_{M_\lambda}(B \in A) = v_{M_\lambda}(A \in A)$. Note that the substitution of B for A is only for the left hand A because (ii) was applied to $A \in A$. By the condition of the lemma, $v_{M_\lambda}(B \in B) = v_{M_\lambda}(B \in A)$ and hence

$v_{M_\lambda}(B\in B)=v_{M_\lambda}(A\in A)$. Similarly for the case when $v_{M_\lambda}(B\in B)$ is 1 or 0, by setting W as $B\in B$. Hence the lemma holds.

Theorem 4.

The Axiom of Extensionality (E) is valid in M_λ.

Proof. We will prove: If $V\in A \leftrightarrow V\in B$ is valid in M_λ then $A\in Z \leftrightarrow B\in Z$ is valid in M_λ. Let $v_{M_\lambda}(A\in C)=1$ or 0. By lemma 8, $A\in C$ has a gen. dep. set D' without any wffs of the form $A\in B'$ for any B' except for A and cases where $B'\in D^S$. Hence the only occurrences of A in D' are of the forms: $A'\in A(A'\neq A)$, $A\in A$ and $A\in B'$ (where $B'\in D^S$). Consider the atomic wff $A\in B'$.

(A) Let $A\in D^S$.

(i) Let $B\in D^S$. Then $z\in A \leftrightarrow z\in B$ is valid in M_0. By the Extensionality Axiom of NBG, $v_{M_0}(B\in B')=v_{M_0}(A\in B')$ and hence $v_{M_\lambda}(B\in B')=v_{M_\lambda}(A\in B')$.

(ii) Let $B\notin D^S$. Then $z\in A \leftrightarrow z\in B$ is valid in M_λ. By the Lukasiewicz logic, $\sim z\in A \vee z\in B . \& . \sim z\in B \vee z\in A$ is valid in M_λ. Hence $v_{M_{\lambda+1}}(B\in B')=v_{M_0}(A\in B')$ and $v_{M_\lambda}(B\in B')=v_{M_\lambda}(A\in B')$.

(B) Let $A\notin D^S$.

(i) Let $\sim z\in A \vee z\in a . \& . \sim z\in a \vee z\in A$ be valid in M_λ for some a. Hence $v_{M_{\lambda+1}}(A\in B')=v_{M_0}(a\in B')$. By the condition of the theorem, $\sim z\in B \vee z\in a . \& . \sim z\in a \vee z\in B$ is valid in M_λ. Hence $v_{M_{\lambda+1}}(B\in B')=v_{M_0}(a\in B')$ and $v_{M_\lambda}(B\in B')=v_{M_\lambda}(A\in B')$.

(ii) Let $(Ax)(Sz)(z\in A \;\&\; \sim z\in x . \vee . z\in x \;\&\; \sim z\in A)$ have the value 1 in M_λ. Hence $v_{M_{\lambda+1}}(A\in B')=0$. [That is, the case (ii) is not a possibility if $v_{M_\lambda}(A\notin B')=1.]$ By the condition of the theorem, (Ax) (Sz)(z∉B & ~z∈x.v.z∈x & ~z∈B) has the value 1 in M_λ. Hence $v_{M_{\lambda+1}}(B\in B')=0$ and $v_{M_\lambda}(B\in B')=v_{M_\lambda}(A\in B')$. Hence, in all cases, $v_{M_\lambda}(B\in B')=v_{M_\lambda}(A\in B')$. By lemma 9, $v_{M_\lambda}(B\in B)=v_{M_\lambda}(A\in A)$. By the condition of the theorem, $v_{M_\lambda}(A'\in B)=v_{M_\lambda}(A'\in A)$. Hence, if Q(A)∈ D', $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By lemma 7, $v_{M_\lambda}(B\in C)=v_{M_\lambda}(A\in C)$. Similarly, if $v_{M_\lambda}(B\in C)=1$ or 0, then $v_{M_\lambda}(A\in C)=v_{M_\lambda}(B\in C)$. Hence the theorem holds.

Theorem 5.

(Az)(z∈x ↔ z∈X)⊃(Aw)(x∈w ↔ X∈w) is valid in M_λ. (I.e. General Axiom 2 is valid in M_λ.)

Proof (i) Let $A\notin D^S$. Let z∈a ↔ z∈A be valid in M_λ. Hence ~z∈ a v z∈A.&.~z∈A v z∈a is valid in M_λ. Hence $v_{M_{\lambda+1}}(A\in c)=v_{M_0}(a\in c)$ and $v_{M_\lambda}(A\in c \leftrightarrow a\in c)=1$, for any $c\in D^S$.

(ii) Let $A\in D^S$. Then, by the Axiom of Extensionality for NBG, the theorem holds.

Theorem 6.

(i) C(x∈y),

(ii) FSCl(X)⊃F(X∈x).

(iii) PSCl(X)⊃P(X∈x),

are all valid in M_λ. (I.e. General Axioms 5, 3 and 4 are valid in M_λ.)

Proof. (i) is valid by definition of M_0. Let $v_{M_\lambda}(FSCl(A))=1$.

Hence $v_{M_\lambda}((Ax)(Sz)(z\in x \ \&\ \sim z\in A . v . z\notin A \ \&\ \sim z\notin x))=1$ and $v_{M_{\lambda+1}}(A\in b)=0$, for any b. Hence $F(A\in x)$ is valid in M_λ.

Let $v_{M_\lambda}(A\notin b)=1$ or 0. Then either $A\in D^S$, $z\in a \leftrightarrow z\in A$ is valid in M_λ for some a, or $(Ax)(Sz)\sim(z\in x \leftrightarrow z\in A)$ is valid in M_λ. Hence $SCl(A)$ has the value 1 or 0 in M_λ. Hence if $v_{M_\lambda}(SCl(A))=\frac{1}{2}$ then $v_{M_\lambda}(A\in b) = \frac{1}{2}$.

Theorem 7.

$(AX)\phi(X)\rightarrow(Ax)\phi(x)$ is valid in M_λ. (I.e. General Axiom 1 is valid in M_λ.)

Proof. Let $v_{M_\lambda}((AX)\phi(X))=1$. Then $v_{M_\lambda}(\phi(X))=1$, for all $X\in D$. Hence $v_{M_\lambda}(\phi(x))=1$, for all $x\in D^S$, since $D^S\subseteq D$. Therefore $v_{M_\lambda}((Ax))=1$.
Let $v_{M_\lambda}((AX)\phi(X))=\frac{1}{2}$. Then $v_{M_\lambda}(\phi(X))=\frac{1}{2}$ or 1, for all X. Hence $v_{M_\lambda}(\phi(x))=\frac{1}{2}$ or 1, for all x. Therefore $v_{M_\lambda}((Ax)\phi(x))=\frac{1}{2}$ or 1.

H nce all the axioms are valid in M_λ. Since D and D^S are sets, the whole proof of M_λ being a model for the axioms can be formalised within Z-F. Since NBG is relatively consistent to Z-F, the above theory is relatively consistent to Z-F.

The above method can be used to extend any set or class theory with a two-valued model with the Axiom of Extensionality to a three-valued class theory satisfying the Axioms of Abstraction and Extensionality. By using appropriate models of NBG, the consistency and independence of the Axiom of Choice, the

Generalised Continuum Hypothesis and the Axiom of Constructibility can be shown. Also the connectives and quantifiers of the three-valued logic used to define standard wffs can be extended to include any which satisfy the following property, P:

(i) <u>For connectives, $\Gamma(p_1, \ldots, p_n)$</u>

Let $v_M(\;(p_1,\ldots,p_n))=1$ (or 0). Let X_0 be the set of indices i such that $v_M(p_i)=0$. Let X_1 be the set of indices i such that $v_M(p_i)$ $=1$. For some structure M', let X_0' be the set of indices i such that $v_M'(q_i)=0$, and let X_1' be the set of indices i such that $v_M'(q_i)=1$. If $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$, then $v_M'(\Gamma(q_1,\ldots,q_n))=1$ (or 0).

(ii) <u>For quantifiers, $(QX)A(X)$</u>

Let $v_M((QX)A(X)=1$ (or 0). Let X_0 be the set of X's in D such that $v_M(A(X))=0$. Let X_1 be the set of X's in D such that $v_M(A(X))$ $=1$. For some structure M', let X_0' be the set of X's in D such that $v_M'(B(X))=0$, and let X_I' be the set of X's in D such that $v_M'(B(X))=1$. If $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$, then $v_M'((QX)B(X))=1$ (or 0).

(iii) <u>For quantifiers, $(Qx)A(x)$.</u>

Similar to (ii), except D^S for D.

<u>Proposition</u>

Any quantifier or connective defined in terms of quantifiers and connectives satisfying the property P also satisfies the property P.

<u>Proof.</u> (i) <u>Connectives.</u>

Let $v_M(\Gamma(\Delta_1(q_1,\ldots,q_n),\ldots,\Delta_m(q_1,\ldots,q_n)))=1$ (or 0), where Γ, Δ_1, $\ldots,\Delta m$ satisfy the property P Let X_0 be the set of indices i such that $v_M(q_i)=0$. Let X_1 be the set of indices i such that $v_M(q_i)=1$. For some structure M', let X_0' be the set of indices i such that $v_M'(r_i)=0$ and let X_I' be the set of indices i such that $v_M'(r_i)=1$. Let $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$.

Let Y_0 be the set of indices i such that $v_M(\Delta_i(q_1,\ldots,q_n))=0$ and let Y_1 be the set of indices i such that $v_M(\Delta_i(q_1,\ldots,q_n))=1$. Let $i \in Y_0 \cup Y_1$. Because $\Delta_i(q_1,\ldots,q_n)$ satisfies the property P, and $X_C \subseteq X_0'$ and $X_1 \subseteq X_1'$, $v_M'(\Delta_i(r_1,\ldots,r_n))=v_M(\Delta_i(q_1,\ldots,q_n))$. Hence, if Y_0' is the set of indices i such that $v_M'(\Delta_i(r_1,\ldots,r_n))=0$ and Y_1' is the set of indices i such that $v_M'(\Delta i(r_1,\ldots,r_n))=1$, then $Y_1 \subseteq Y_1'$ and $Y_0 \subseteq Y_0'$. Since $\Gamma(\Delta_1,\ldots,\Delta_m)$ satisfies the property P, $v_M'(\Gamma(\Delta_1,\ldots,\Delta_m))=1$ (or 0).

(ii) Quantifiers.

Let $v_M(\Gamma((QX)\Delta(A(X))))=1$ (or 0), where Γ, Δ and (QX) satisfy the property P. Let X_0 be the set of X's from D such that $v_M(A(X))=0$. Let X_1 be the set of X's from D such that $v_M(A(X))=1$. For some structure of M', let X_0' be the set of X's from D such that $v_M'(B(X))=0$, and let X_1' be the set of X's from D such that $v_M'(B(X))=1$. Let $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$.

Because Δ satisfies the property P, if $v_M(\Delta(A(X)))=1$ or 0 then $v_M'(\Delta(B(X)))=v_M(\Delta(A(X)))$, for any $X \in D$. If Y_0 is the set of X's in D such that $v_M(\Delta(A(X)))=0$, Y_1 is the set of X's in D such that $v_M(\Delta(A(X)))=1$, Y_0' is the set of X's in D such that $v_M'(\Delta(B(X)))=0$, and Y_1' is the set of X's in D such that $v_M'(\Delta(B(X)))=1$, then $Y_0 \subseteq Y_0'$ and $Y_1 \subseteq Y_1'$. Because (QX) satisfies the property P, if $v_M((QX)\Delta(A(X)))=1$ or 0 then $v_M'((QX)\Delta(B(X)))=v_M((QX)\Delta(A(X)))$. Since Γ satisfies the property P, $v_M'(\ ((QX)\Delta(B(X)))=1$ (or 0). Similarly for quantifiers, (Qx).

Some examples of connectives satisfying the property P are:

	1	0	$\frac{1}{2}$
1	1, 0 or $\frac{1}{2}$	1, 0 or $\frac{1}{2}$	$\frac{1}{2}$
0	1, 0 or $\frac{1}{2}$	1, 0 or $\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

	1	0	$\frac{1}{2}$
1	1	1, 0 or $\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1 or $\frac{1}{2}$
$\frac{1}{2}$	1 or $\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

1	1
0	1
$\frac{1}{2}$	1 or $\frac{1}{2}$

1	1, 0 or $\frac{1}{2}$
0	1, 0 or $\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$

The Lukasiewicz (AX), (Ax), (SX), (Sx) are examples satisfying the quantifier property P.

The following connectives are <u>not</u> examples :

$\rightarrow$	1	0	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$
0	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	1

$\leftrightarrow$	1	0	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$
0	0	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1

$\supset$	1	0	$\frac{1}{2}$
1	1	0	$\frac{1}{2}$
0	1	1	1
$\frac{1}{2}$	1	1	1

T	
1	1
0	0
$\frac{1}{2}$	0

C	
1	1
0	1
$\frac{1}{2}$	0

To show that any of the connectives or quantifiers satisfying the property P can be used to define standard wffs and hence be substituted into the Abstraction Axiom, it is only necessary to examine lemmas 1 and 4 in the proof. Lemma 1 is obvious from the definition of the property P. In lemma 4, leave out the original steps for the connectives and quantifiers and replace it by the following:

(i) Let $P(A)$ be $\Gamma(R_1(A),\ldots,R_n(A))$ and let $v_{M_\lambda}(\Gamma(R_1(A),\ldots,R_n(A)))=1$ or 0. Let $\Gamma(R_1(A),\ldots,R_n(A))$ be W. Then $v_{M_{\nu_W}}(W)=1$ or 0. Let X_0 be the set of indices i such that $v_{M_{\nu_W}}(R_i(A))=0$ and let X_1 be the set of indices i such that $v_{M_{\nu_W}}(R_i(A))=1$. Since $\nu_{R_i(A)} \leq \nu_W$, for all $i \in X_0 \cup X_1$, $D(R_i(A)) \subseteq D(W)$, for all $i \in X_0 \cup X_1$. By the lemma condition, for each $Q(A) \in D(R_i(A))$ where $i \in X_0 \cup X_1$, $v_{M_\lambda}(Q(B)) = v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R_i B)) = v_{M_\lambda}(R_i(A))$, for all $i \in X_0 \cup X_1$. Let X_0' be the set of indices i such that $v_{M_\lambda}(R_i(B))=0$ and let X_1' be the set of indices i such that $v_{M_\lambda}(R_i(B))=1$. Hence $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. By the property of Γ, $v_{M_\lambda}(\Gamma(R_1(B),\ldots,R_n(B))) = v_{M_{\nu_W}}(\Gamma(R_1(A),\ldots,R_n(A)))$ and $v_{M_\lambda}(P(B)) = v_{M_\lambda}(P(A))$.

(ii) Let $P(A)$ be $(QZ)R(A,Z)$ and $v_{M_\lambda}((QZ)R(A,Z))=1$ or 0. Let $(QZ)R(A,Z)$ be W. Let X_0 be the set of Z's in D such that $v_{M_{\nu_W}}(R(A,Z))=0$ and let X_1 be the set of Z's in D such that $v_{M_{\nu_W}}(R(A,Z))=1$. Since $\nu_{R(A,Z)} \leq \nu_W$, for all $Z \in X_0 \cup X_1$, $D(R(A,Z)) \subseteq D(W)$, for all $Z \in X_0 \cup X_1$. By the lemma condition, for each $Q(A) \in D(R(A,Z))$, $v_{M_\lambda}(Q(B)) = v_{M_\lambda}(Q(A))$, where $Z \in X_0 \cup X_1$. By the ind. hyp., $v_{M_\lambda}(R(B,Z)) = v_{M_\lambda}(R(A,Z))$, for all $Z \in X_0 \cup X_1$. Let X_0' be the set of all Z's in

D such that $v_{M_\lambda}(R(B,Z))=0$ and let X_1' be the set of all Z's in D such that $v_{M_\lambda}(R(B,Z))=1$. Hence $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. By the property of (QZ), $v_{M_\lambda}((QZ)R(B,Z))=v_{M_{\nu_W}}((QZ)R(A,Z))$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iii) Let P(A) be (Qz)R(A,z). This case follows as for (ii) except that D^S replaces D.

There is a further generalisation which allows any set or class theory using a many-valued (finite or infinite) logic L and with a denumerable model in which the Axiom of Extensionality is satisfied to be extended to a class theory, using a logic L' of one more value and with a model in which the Axioms of Extensionality and Abstraction are satisfied.

The many-valued logic L must contain a quantifier S such that (SZ)A(Z) takes the value m (where m is some designated value) iff at least one of the A(Z)'s are designated and takes the value n (where n is some undesignated value) iff all of the A(Z)'s are undesignated (similarly for (Sz)A(z)), a quantifier A such that (AZ)A(Z) takes the value m (same value as above) iff all of the A(Z)'s are designated and takes the value n (same value as above) iff at least one of the A(Z)'s is undesignated (similarly for (Az)A(z)), and equivalence connective $\leftrightarrow$ such that $p \leftrightarrow q$ is designated iff p and q take the same value, and an implication connective $\supset$ such that $p \supset q$ is designated iff q is designated or p is undesignated.

The many-valued logic L', which has an extra value (call it pd) added to L, must contain appropriate extensions of S, A, $\leftrightarrow$ and $\supset$. The value pd is undesignated. $p \supset q$ is defined so that it is designated iff q is designated or p is undesignated. $p \leftrightarrow q$ is defined so that if p and q take values in L, then $p \leftrightarrow q$ takes the value in L for its value in L', if p does not take the value pd and q takes the value pd or if p takes the value pd and q does not then $p \leftrightarrow q$ takes the value pd, and if p and q both take the value pd then $p \leftrightarrow q$ is designated. The quantifier S is defined in in L' as follows: If A(Z) has a designated value for some Z, then (SZ)A(Z) has the value m. If A(Z) has an undesignated value, not pd, for all Z then (SZ)A(Z) has the value n. Otherwise (SZ)A(Z) has the value pd. The quantifier A is defined in L' as follows: If A(Z) has a designated value for all Z, then (AZ)A(Z) has the value m. If A(Z) has an undesignated value, not pd, for some Z, then (AZ)A(Z) has the value n. Otherwise (AZ)A(Z) has the value pd. Similarly for (Az)A(z) and (Sz)A(z).

The Axiom of Extensionality can now be stated as $(AZ)(Z \in X \leftrightarrow Z \in Y) \supset (AZ)(X \in Z \leftrightarrow Y \in Z)$. The Abstraction Axiom can be stated as (SY)(AX) $(X \in Y \leftrightarrow \emptyset(X, z_1, \ldots, z_m, Z_1, \ldots, Z_n))$, where $\emptyset$ is constructed from atomic wffs $U \in V$, $U \in v$, $u \in V$, $u \in v$, using the connectives and quantifiers used in forming standard wffs.

The Axiom of Extensionality for special classes can be stated as $(\Lambda z)(z\in x\leftrightarrow z\in y)\supset(\Lambda z)(x\in z\leftrightarrow y\in z)$. $SCl(X)$ is defined as $(Sx)(\Lambda z)$ $(z\in x\leftrightarrow z\in X)$.

The propositional constants are left out from the atomic wffs and if atomic wffs with some of these values are wanted then perhaps an atomic wff of the form $a\in b$ can be used.

The connectives and quantifiers used in forming standard wffs are ones which satisfy the property S:

(i) <u>For connectives $\Gamma(p_1,\ldots,p_n)$.</u>

Let $v_M(\Gamma(p_1,\ldots,p_n))=k$, some value of L. Let X_m be the set of indices i such that $v_M(p_i)=m$, for each value m of L. For some structure M', let X'_m be the set of indices i such that $v_{M'}(q_i)=m$, for each value m of L. If $X_m\subseteq X_m'$, for all m in L, then $v_{M'}(\Gamma(q_1,\ldots,q_h))=k$.

(ii) <u>For quantifiers $(QX)A(X)$.</u>

Let $v_M((QX)A(X))=k$, some value of L. Let X_m be the set of X's in D such that $v_M(A(X))=m$, for each value m of L. For some structure M', let X_m' be the set of X's in D such that $v_M'(B(X))=m$, for each m in L. If $X_m\subseteq X_m'$, for all m of L, then $v_M'((QX)B(X))=k$.

(iii) <u>For quantifiers $(Qx)A(x)$.</u>

Similar to (ii) except D^S for D.

Using a similar proof to that used for the three-valued case, it can be shown that any quantifier or connective defined in terms of quantifiers and connectives satisfying the property S also satisfies the property S.

Note that the quantifiers S and A of L' satisfy the property S For the definition of $M_1 \leq M_2$, for two structures M_1 and M_2 the generalisation is as follows: $M_1 \leq M_2$ iff, for any atomic wff P, if $v_{M_1}(P)=m$, for some value m of L, then $v_{M_2}(P)=m$. Lemma 1 follows by the property S for connectives and quantifiers used in forming standard wffs. It takes the form: Let $M \leq M'$, where M and M' are two structures on D. Then, for any standard wff P, if $v_M(P)=m$, for some value m of L, then $v_{M'}(P)=m$.

Define the structure M_0 as follows:

If $A \notin D^S$ or $B \notin D^S$, then $v_{M_0}(A \in B)=pd$. If $A \in D^S$ and $B \in D^S$, then $v_{M_0}(A \in B)=$ the value of L given to $A \in B$ in the model of the special class theory.

Assuming M_μ defined for some ordinal , $M_{\mu+1}$ is defined as follows:
For all standard wffs P, $v_{M_{\mu+1}}(A \in \{X : P(X)\})=v_{M_\mu}(P(A))$.
If $v_{M_\mu}(z \in A)=v_{M_\mu}(z \in a)$ for all $z \in D^S$, for some $a \in D^S$, then $v_{M_{\mu+1}}(A \in b)=v_{M_0}(a \in b)$. If there is no $a \in D^S$ such that for all $z \in D^S$, $v_{M_\mu}(z \in A)=v_{M_\mu}(z \in a)$, then $v_{M_{\mu+1}}(A \in b)=v_{M_\mu}(SCl(A))$.
Note that SCl(X) satisfies the property S, because $z \in x$ only takes values in L. Also $v_{M_\mu}(z \in A)=v_{M_\mu}(z \in a)$ for all $z \in D^S$, for some $a \in D^S$, iff $v_{M_\mu}(SCl(A))$ is designated, and there is no $a \in D^S$ such that for all $z \in D^S$, $v_{M_\mu}(z \in A)=v_{M_\mu}(z \in a)$ iff $v_{M_\mu}(SCl(A))$ is undesignated.

If μ is a limit ordinal, on the assumption that $M_\nu \leq M_\tau$ for all $\nu \leq \tau$, for all $\tau < \mu$, for all atomic wffs P, if $v_{M_\nu}(P)=k$, for some value k of L, for some $\nu<\mu$, then $v_{M_\mu}(P)=k$, and if $v_{M_\nu}(P)=pd$ for all $\nu<\mu$, then $v_{M_\mu}(P)=pd$.

Lemma 2 follows similarly to before. In case (B), let $v_{M_\nu}(A\in b)=k$, for some value k of L. Then there is an ordinal $\eta<\nu$ such that $v_{M_\eta}(SCl(A))=\ell$, for some value ℓ of L. Since $\eta\leq\mu-1$, $M_\eta\leq M_{\mu-1}$. Hence $v_{M_{\mu-1}}(SCl(A))=\ell$. If ℓ is undesignated, $\ell=k$ and $v_{M_\mu}(A\in b)=k$. If ℓ is designated, then there is an $a\in D^S$ such that $v_{M_0}(z\in a)=v_{M_\eta}(z\in A)$ for all $z\in D^S$. Then $v_{M_0}(a\in b)=k$. Hence $v_{M_{\mu-1}}(z\in A)=v_{M_0}(z\in a)$, for all $z\in D^S$, and $v_{M_\mu}(A\in b)=k$.

Lemma 3 follows similarly to before except that there is one increasing chain of subsets of the denumerable set of all atomic wffs for every value of L. Theorems 1, 2 and 3 follow similarly to before. The definitions of ν_P and dependent set D(P) are the same except that all values of L must be put in place of values 1 and 0.

Lemma 4 can be shown for connectives and quantifiers satisfying the property S by a simple generalisation using X_k, where k ranges over the values of L, instead of using X_0 and X_1.

Corresponding standard wff and general dependent set are defined similarly. Lemmas 5, 6, 7 and 8 follow as before with the values of L in place of 1 and 0. In lemma 9, (ii) (A) becomes: Let SCl(A) be valid in M_λ. Then $v_{M_\lambda}(z\in A)=v_{M_\lambda}(z\in a)$, for all $z\in D^S$, for some $a\in D^S$. Hence $v_{M_{\lambda+1}}(A\in B')=v_{M_0}(a\in B')$. By the condition of the

$v_{M_\lambda}(z \in B) = v_{M_\lambda}(z \in A)$, for all $z \in D^S_\lambda$. Hence $v_{M_{\lambda+1}}(B \in B') = v_{M_0}(A \in B')$. Hence lemma, $v_{M_\lambda}(A \in B') = v_{M_\lambda}(B \in B')$. (ii) (B) becomes: Let SCl(A) be invalid in M_λ. $v_{M_\lambda}(SCl(A)) \neq pd$ because $v_{M_\lambda}(A \in B')$ is a value of L. Hence $v_{M_{\lambda+1}}(A \in B') = v_{M_\lambda}(SCl(A))$. By the lemma condition, $v_{M_\lambda}(SCl(A)) = v_{M_\lambda}(SCl(B))$. Hence $v_{M_{\lambda+1}}(B \in B') = v_{M_\lambda}(SCl(A))$ and $v_{M_\lambda}(B \in B') = v_{M_\lambda}(A \in B')$. The rest of lemma 9 follows as before.

In theorem 4, (B) (i) and (B) (ii) are similar to (ii) (A) and (ii) (B), resp., of lemma 9. Otherwise the theorem follows as before.

Theorem 5 follows as before.

Theorem 6 needs two monadic operators: C such that Cp is designated iff p takes a value in L, and U such that Up is designated iff p is undesignated. Theorem 6 becomes: (i) $C(x \in y)$ and (ii) $USCl(X) \supset SCl(X) \leftrightarrow X \in x$, are valid in M_λ, both of which are obvious.

Theorem 7 becomes: $(AX)Q(X) \supset (Ax)Q(x)$ is valid in M_λ, which is obvious. Hence M_λ is a model for the class theory with logic L' and with generalisations of the previous three-valued axioms. This class theory is relatively consistent to Z-F and the class theory with logic L.

There are certain advantages the above method of avoiding the class paradoxes has. It allows each predicate (with some restriction on the connectives) to generate a class and separates the "paradoxical" class membership statements from the "non-paradoxical" ones using a criterion of circularity of definition. For, in order for a membership statement to take the value $\frac{1}{2}$ (or pd), there must be some circularity involved, in the sense that its value is dependent on itself or on the values of membership statements whose values are dependent on themselves. If there is no such circularity then there is a chain of dependent membership statements leading from the membership statement in question right back to membership statements of NBG (or other model) and propositional constants ($\neq \frac{1}{2}$ (or pd)). This is represented in the proof by the general dependent sets of a membership statement. If there is such a chain of dependent membership statements then the membership statement in question takes a value $\neq \frac{1}{2}$ (or pd).

By basing the system on NBG, this allows the whole of Mathematics to be deduced in the usual two-valued logic. One can make true and false statements about the universal class and Russell class which cannot normally be made in other attempts to avoid the class paradoxes. One can make true and false statements about classes of proper classes which cannot be made in NBG.

By defining all classes which can be generated by a predicate we can get a broader picture and see how the paradoxes arise; indeed, it shows that it is the circularity of definition of certain membership statements that leads more directly to the paradoxes than just the inclusion of a certain range of classes because true and false statements can be made about these classes and further, by using some general criterion for the rejection of classes, one may well reject classes which lead to no paradoxes at all.

CHAPTER 7.

A 4-VALUED CLASS THEORY.

In Chapters 4 and 6, there ~~was~~ were presented two 3-valued class theories, one consisting of a class theory like NBG but containing a theory of individuals as well and using a 3-valued significance logic, and the other consisting of an extension of NBG to a class theory satisfying the Axioms of Abstraction and Extensionality and using a 3-valued Lukasiewicz logic. In this chapter, I want to combine these two theories into a class theory using a 4-valued logic. This is done by taking the classes of Chapter 4 as special classes and extending this special class theory to a 4-valued class theory by adding the Axioms of Abstraction and Extensionality, similar to the extension of NBG to a 3-valued class theory as it appears in Chapter 6. The purpose of constructing this 4-valued theory is to obtain an all-embracing theory involving all classes and individuals. We can form the class of all classes, the class of all individuals and the class of all classes and individuals, i.e. the class of all "things".

The formalisation of the theory is as follows :

Primitives.

1. U,V,W,X,Y,Z,......(variables over classes and individuals.)
2. u,v,w,x,y,z,.......(variables over special classes, i.e. the classes of Chapter 4, ~~)~~ and individuals.
3. $\in$ (is a member of), o(overlaps).
4. $\sim, \&, \supset, T_n$ (connectives of the 4-valued logic.)

5. A,S (quantifiers of the 4-valued logic.)

Formation Rules.

1. For variables, X,Y,x,y, the following are atomic wffs : XoY, Xox, xoX, xoy, $X\in Y$, $X\in x$, $x\in X$, $x\in y$.

2. The propositional constants $1, 0, \frac{1}{2}$ are atomic wffs.

3. If B and C are wffs and x and X are variables then $\sim B$, $B \& C$, $B \supset C$, $T_n B$, $(AX)B$, $(SX)B$, $(Ax)B$, $(Sx)B$ are wffs.

Definitions.

$Cl(X) =_{df} (SY)S(Y\in X)$. (X is a class.)

$I(X) =_{df} \sim Cl(X)$. (X is an individual.)

$Cl_s(x) =_{df} (Sy)S(y\in x)$. (x is a special class.)

$I_s(x) =_{df} \sim Cl(x)$. (x is an individual.)

$(Ak)\phi(k) =_{df} (AX)(I(X) \supset \phi(X))$.

$(Sk)\phi(k) =_{df} (SX)(T_n I(X) \& \phi(X)$.

k,l,m,n,........(variables over individuals.)

$(AF)\phi(F) =_{df} (AX)(Cl(X) \supset \phi(X))$.

$(SF)\phi(F) =_{df} (SX)(T_n Cl(X) \& \phi(X))$.

F,G,H,I,J,.......(variables over classes.)

$(Af)\phi(f) =_{df} (Ax)(Cl_s(x) \supset \phi(x))$.

$(Sf)\phi(f) =_{df} (Sx)(T_n Cl_s(x) \& \phi(x))$.

f,g,h,i,j,.......(variables over special classes.)

$M(f) =_{df} (Sg)(f\in g)$. (f is a set.)

$(Af')\phi(f') =_{df} (Af)(M(f) \supset \phi(f))$.

$(Sf')\phi(f') =_{df} (Sf)(T_n M(f) \& \phi(f))$.

$f', g', h', i', j', \ldots\ldots\ldots$ (variables over sets.)

$(Au')\phi(u') =_{df} (Au)(M(u) \vee I_s(u) \supset \phi(u))$.

$(Su')\phi(u') =_{df} (Su)(T_n(M(u) \vee I_s(u)) \;\&\; \phi(u))$.

$u', v', w', x', y', z', \ldots\ldots\ldots$ (variables over sets and individuals.)

$X=Y =_{df} (Ak)(koX \dot{\leftrightarrow} koY) \vee (AZ)(Z\epsilon X \dot{\leftrightarrow} Z\epsilon Y)$. (X is identical with Y.)

$V(X) =_{df} (AZ)C(Z\epsilon X)$.

The definitions of the theory of classes and individuals of Chapter 4. [Each such definition is distinguished from any similar definition for classes (and individuals) in general by the symbol 's'. For example : $x \overset{s}{=} y =_{df} (Ak)(kox \dot{\leftrightarrow} koy) \vee (Az)(z\epsilon x \dot{\leftrightarrow} z\epsilon y)$.]

$SCl(F) =_{df} (Sf)(Au)(u\epsilon f \leftrightarrow u\epsilon F)$. (F is a special class in that it has the same special class members as some special class but F may not lie in the range of the special class variables.)

<u>General Axioms.</u>

1. $(AX)\phi(X) \rightarrow (Ax)\phi(x)$.
2. $(Az)(z\epsilon f \leftrightarrow z\epsilon F) \supset (Ag)(f\epsilon g \leftrightarrow F\epsilon g)$.
3. $FSCl(F) \supset F(F\epsilon f)$.
4. $PSCl(F) \supset P(F\epsilon f)$.
5. $Cl(X) \supset S(Y\epsilon X)$.
6. $C(x\epsilon f)$.
7. $C(kol)$.
8. $Cl(X) \vee Cl(Y) \supset {\sim}S(XoY)$.
9. $k=l \supset . k\epsilon F \leftrightarrow l\epsilon F$.

<u>Individual Axioms.</u>

1. $kol \equiv (Sm)(An)(nom \supset nok \;\&\; nol)$.

2. $(Sk)(k \in f') \supset (Sl)(lFuf')$.

3. $(Sf')((Ak)(k \in f' \equiv \emptyset(k, l_1, \ldots, l_m)) \;\&\; (Ag')(g' \in f'))$, where $\emptyset$ is constructed using only o, ~, &, A and variables quantified over individuals.

4. $k=l \supset . k \in f \equiv l \in f$.

5. $(Sx)I(x)$.

Special Class Axioms.

T. $f \overset{s}{=} g \supset (Ah)(f \in h \equiv g \in h)$.

P. $(Ax')(Ay')(Sf')(Au')(u' \in f' \equiv T(u' \overset{s}{=} x' \vee u' \overset{s}{=} y'))$.

N. $(Sf')(Ax')(\sim x' \in f')$.

U. $(Af')(Sg')(Ax')(x' \in g' \equiv (Sh')(x' \in h' \;\&\; h' \in f'))$.

W. $(Af')(Sg')(Ax')(x' \in g' \equiv T(Ay')(y' \in x' \supset y' \in f'))$.

S. $(Af')(Ag)(Sg')(Ax')(x' \in g' \equiv \quad x' \in f' \;\&\; x' \in g)$.

B. $(Ax_1', \ldots, x_k', y_1, \ldots, y_m) S\emptyset(x_1', \ldots, x_k', y_1, \ldots, y_m) \supset (Sf)(Ax_1', \ldots, x_k')(\langle x_1', \ldots, x_k' \rangle_s \in f \equiv \emptyset(x_1', \ldots, x_k', y_1, \ldots, y_m))$, where $\emptyset$ is constructed using o, $\in$, ~, $\supset$, T_n, A, S(quantifier) such that only variables over sets and individuals are quantified, and $x_1', \ldots, x_k'$, $y_1, \ldots, y_m$ are all the free variables of $\emptyset$ and f is not amongst them.

R. $(Af')(Un(f) \supset (Sg')(Ax')(x' \in g' \equiv (Sy')(\langle y', x' \rangle_s \in f \;\&\; y' \in f')))$.

I. $(Sf')(O \in f' \;\&\; (Ag')(g' \in f' \supset g' \cup \{g'\}_s \in f'))$.

The Axioms of Choice, Restriction, Constructibility and the Generalised Continuum Hypothesis can be added if one wishes. As will be shown later, it does not affect the consistency proof of the 4-valued theory whether they are put in or omitted.

Class Axioms.

A. $(AU, V_1, \ldots, V_i, x_1, \ldots, x_k) S\emptyset(U, V_1, \ldots, V_i, x_1, \ldots, x_k) \supset (SF)(AU)(U \in F \leftrightarrow \emptyset(U, V_1, \ldots, V_i, x_1, \ldots, x_k))$, where $\emptyset$ is either a propositional constant or constructed using o, $\in$, ~, &, v, S(sig.), A, S(quantifier), where quantification is unrestricted using variables of type x or X or restricted to a predicate A(x) or A(X), in which every occurrence of a variable over classes and individuals is covered by S(sig.), and where A is constructed from atomic wffs using only ~, &, v, S (sig.), A, S(some), where the quantifiers A and S are unrestricted.

E. $F = G \supset (AH)(F \in H \leftrightarrow G \in H)$.

The only theorems we will deal with are certain ones following from the General Axioms as the others can be found in Mendelson or in Chapters 4, 5, or 6.

T.1. $\underline{(Sx)\emptyset(x) \rightarrow (SX)\emptyset(X)}$.

I . $\sim S(Sx)\emptyset(x) \supset T((Sx)\emptyset(x) \rightarrow (SX)\emptyset(X)$ ________ (1)

Hyp : $S(Sx)\emptyset(x)$ ________ (2)

Hyp : $\sim S(SX)\emptyset(X)$ ________ (3)

(3) : $(AX)\sim S\emptyset(X)$ ________ (4)

(4), Gen. Ax. 1 : $(Ax)\sim S\emptyset(x)$ ______ (5)

(5) : $\sim S(Sx)\emptyset(x)$ ______ (6)

(2),(3),(6) : $S(SX)\emptyset(X)$ ______ (7)

Hyp : $\sim T(SX)\emptyset(X)$ ______ (8)

(8) : $(AX)\sim T\emptyset(X)$ ______ (9)

(9), Gen.Ax.1 : $(Ax)\sim T\emptyset(x)$ ______(10)

(10) : $\sim T(Sx)\emptyset(x)$ ______(11)

(8),(11) : $T(Sx)\emptyset(x) \supset T(SX)\emptyset(X)$ ______(12)

Hyp : $F(SX)\emptyset(X)$ ______(13)

(13) : $(AX)(F\emptyset(X) \vee \sim S\emptyset(X))$ ______(14)

(14), Gen.Ax.1 : $(Ax)(F\emptyset(x) \vee \sim S\emptyset(x))$ ______(15)

(2), (15) : $F(Sx)\emptyset(x)$ ______ (16)

(2), (7), (13), (16) : $T(Sx)\emptyset(x) \vee F(Sx)\emptyset(x) \supset T(SX)\emptyset(X)$ ______(17)

(7) : $F(Sx)\emptyset(x) \supset S(SX)\emptyset(X)$ ______(18)

(2), (7), (12), (17), (18) : $S(Sx)\emptyset(x) \supset T((Sx)\emptyset(x) \rightarrow (SX)\emptyset(X))$ ______(19)

(1), (19) : T.1.

T.2. $\underline{Cl_s(x) \leftrightarrow Cl(x)}$.

T.1, Defns.Cl_s,Cl : $Cl_s(x) \rightarrow Cl(x)$ ______(1)

Hyp : $Cl(x)$ ______(2)

(2), Gen. Axs. 1, 5 : $S(y \notin x)$ ______(3)

(3) : $(Sy)S(y \notin x)$ ______(4)

(4), Defn. Cl_s : $Cl_s(x)$ ______(5)

(2), (5) : $Cl(x) \rightarrow Cl_s(x)$ ______(6)

(1), (6) : T.2.

T.3. $\underline{I_s(x) \leftrightarrow I(x)}$.

T.2, Defns. I_s, I : T.3.

T.4. $\underline{x \overset{s}{=} y \supset x = y}$.

Hyp : $x \overset{s}{=} y$ ______(1)

(1), Defn. $\overset{s}{=}$: $(Ak)(kox \leftrightarrow koy) \vee (Az)(z \in x \leftrightarrow z \in y)$ ______(2)

Hyp : $I(x) \& I(y)$ ______(3)

(2), (3) : $(Ak)(kox \leftrightarrow koy)$ _____(4)

(3), (4) : $(Ak)(kox \leftrightarrow koy) \vee (AZ)(Z\epsilon x \leftrightarrow Z\epsilon y)$ ______(5)

(5), Defn.= : $x=y$ ______(6)

(3), (6) : $I(x) \& I(y) \supset x=y$ ______(7)

Hyp : $I(x) \& Cl(y)$ ______(8)

(1), (2), (8) : $\sim S(x \overset{s}{=} y)$ ______(9)

(2), (8), (9) : $\sim T(I(x) \& Cl(y))$ _____(10)

Similarly, $\sim T(Cl(x) \& I(y))$ _______(11)

Hyp : $Cl(x) \& Cl(y)$ ______(12)

(1), (12), Ax.T : $(Ah)(x\epsilon h \equiv y\epsilon h)$ _____(13)

Hyp : $TSCl(F)$ ______(14)

(14), Defn. SCl : $(Aw)(w\epsilon F \leftrightarrow w\epsilon f_1)$ _____(15) (f_1 is a constant.)

(15), Gen. Ax. 2 : $(Ag)(f_1\epsilon g \leftrightarrow F\epsilon g)$ _____(16)

(16) : $F\epsilon x \leftrightarrow f_1\epsilon x$ ______(17)

(16) : $F\epsilon y \leftrightarrow f_1\epsilon y$ ______(18)

(2) : $f_1\epsilon x \leftrightarrow f_1\epsilon y$ ______(19)

(17), (18), (19) : $F\epsilon x \leftrightarrow F\epsilon y$ _____(20)

(14), (20) : $TSCl(F) \supset F\epsilon x \leftrightarrow F\epsilon y$ _____(21)

Hyp : $FSCl(F)$ _____(22)

(22), Gen. Ax. 3 : $F(F\epsilon x) \& F(F\epsilon y)$ _____(23)

(23) : $F\epsilon x \leftrightarrow F\epsilon y$ _______(24)

(22), (24) : $FSCl(F) \supset . F\epsilon x \leftrightarrow F\epsilon y$ _____(25)

Hyp : $PSCl(F)$ _____(26)

(26), Gen. Ax. 4 : $P(F\ x)$ & $P(F\ y)$ ____(27)

(27) : $F \epsilon x \leftrightarrow F \epsilon y$ ______(28)

(26), (28) : $PSCl(F) \supset . F \epsilon x \leftrightarrow F \epsilon y$ _____(29)

(21), (25), (29) : $F \epsilon x \leftrightarrow F \epsilon y$ ______(30)

(2), (12) : $(Ak)(k \epsilon x \leftrightarrow k \epsilon y)$ ______(31)

(2), (12), (30), (31) : $x=y$ _______(32)

(12), (32) : $Cl(x)$ & $Cl(y) \supset x=y$ ______(33)

(7), (10), (11), (33) : $x=y$ _____(34)

(1), (34) : T.4.

T.5. <u>$x \overset{s}{=} y \equiv x=y$.</u>

Gen. Ax. 1 : $(AZ)(Z \epsilon x \leftrightarrow Z \epsilon y) \supset (Az)(z \epsilon x \leftrightarrow z \epsilon y)$ _____(1)

(1), Defns. $\overset{s}{=}$, $=$: $x=y \supset x \overset{s}{=} y$ ______(2)

(2), T.4 : T.5.

T.6. <u>$PSCl(F) \equiv P(F \epsilon f)$.</u>

Hyp : $P(F \epsilon f)$ ______(1)

(1), Gen.Ax.3 : $\sim FSCl(F)$ _______(2)

Hyp : $TSCl(F)$ _____(3)

(3), Defn.SCl : $(Az)(z \epsilon f_1 \leftrightarrow z \epsilon F)$ ______(4) (f_1 is a constant.)

(4), Gen.Ax.2 : $(Ag)(f_1 \epsilon g \leftrightarrow F \epsilon g)$ _____(5)

(5) : $C(F \epsilon g)$ ______(6)

(1), (3), (6) : $\sim TSCl(F)$ ______(7)

(2), (7) : $PSCl(F)$ _______(8)

(1), (8) : $P(F \epsilon f) \supset PSCl(F)$ _______(9)

(9), Gen. Ax. 4 : T.6.

T.7. $\underline{x \overset{s}{=} y \leftrightarrow x=y.}$

Hyp : $P(x=y)$ _____(1)

(1), Defn. = : $Cl(x)$ & $Cl(y)$ & $P(AZ)(Z \in x \leftrightarrow Z \in y)$ _______(2)

(2) : $P(Z_1 \in x \leftrightarrow Z_1 \in y)$ ______(3) (Z_1 is a constant.)

(3) : $P(Z_1 \in x)$ v $P(Z_1 \in y)$ _______(4)

(4), T.6 : $PSCl(Z_1)$ _______(5)

(5), Gen. Ax. 4 : $P(Z_1 \in x)$ & $P(Z_1 \in y)$______(6)

(6) : $T(Z_1 \in x \leftrightarrow Z_1 \in y)$ _____(7)

(1), (3), (7) : $\sim P(x=y)$ ______(8)

Defns. $\overset{s}{=}$, = : $S(x \overset{s}{=} y) \equiv S(x=y)$ ______(9)

(8), (9), T.5 : T.7.

T.8. $\underline{TSCl(f)}$.

Defn. SCl : $SCl(F) \leftrightarrow (Sg)(Az)(z \in g \leftrightarrow z \in F)$ _____(1)

(1), Gen. Ax. 1 : $SCl(f) \leftrightarrow (Sg)(Az)(z \in g \leftrightarrow z \in f)$ ______(2)

(2) : $TSCl(f)$.

T 9. $\underline{PSCl(H)}$.

Defn. SCl : $SCl(H) \leftrightarrow (Sg)(Az)(z \in g \leftrightarrow z \in H)$ _____(1)

Defn. H : $\sim F(z \in g \leftrightarrow z \in H)$ _____(2)

(2) : $P(z \in g \leftrightarrow z \in H)$ ______(3)

(3) : $P(Sg)(Az)(z \in g \leftrightarrow z \in H)$ ______(4)

(1), (4) : $PSCl(H)$.

T.10. $\underline{P(H \in f)}$.

T.9, Gen. Ax. 4 : T.10.

T.11. $\underline{\sim V(f)}$.

T.10, Defn. V : T.11.

The theorems of Chapter 5 follow in this theory with some modifications due to the inability in the Abstraction Axiom of distinguishing classes and individuals. The results on Boolean operations follow with little change. However, for the example, $(SF)(AX)(X\epsilon F \leftrightarrow Z\epsilon X)$, $Z\epsilon X$ may be non-significant and there is no way of restricting the variable X to classes only. The usual method is to use the predicate, $Cl(X)$ & $Z\epsilon X$, but this is non-significant when X is an individual. Although restricted quantification is allowed in the Abstraction Axiom, it cannot be used for the variable X. One could try the predicate $(SF)(F=X$ & $Z\epsilon F)$, but then $F=X$ would have to be represented as : $(Ak)(\sim koF \vee koX.\&.\sim koX \vee koF) \vee (AZ)(\sim Z\epsilon F \vee Z\epsilon X.\&.\sim Z\epsilon X \vee Z\epsilon F)$. This is false when X is an individual, which is what one wants, but it is not true when X is a class such that $\sim V(X)$.

For an individual, X, $\{X\}$ can be defined as the unique class Y such that $(AU)(U\epsilon Y \leftrightarrow (Ak)(\sim koU \vee koX.\&.\sim koX \vee koU))$. If U is a class, then $U\epsilon Y$ is false. If U is an individual, then $U\epsilon Y$ iff $U=X$.

For a class, X, $\{X\}$ can be defined as the unique class Y such that $(AU)(U\epsilon Y \leftrightarrow (AZ)(\sim Z\epsilon U \vee Z\epsilon X.\&.\sim Z\epsilon X \vee Z\epsilon U))$. If U is an individual, then $U\epsilon Y$ is not true. If U is a class then $U\epsilon Y$ iff $U=X$, if $V(X)$ holds.

So similar problems to those encountered in Chapter 5 will again appear in the case of $\{X\}$, and similarly for $\mathcal{P}(X)$.

As in the previous chapter, we will introduce the notion of absoluteness for functions and wffs. f is absolute if $f_s(x_1,\ldots,x_n)=f(x_1,\ldots,x_n)$, for all $x_1,\ldots,x_n$, where f_s is defined in the same way as f except that all quantification is with special class variables. $\emptyset$ is absolute if $\emptyset_s(x_1,\ldots,x_n) \leftrightarrow \emptyset(x_1,\ldots,x_n)$, for all $x_1,\ldots,x_n$, where $\emptyset_s$ is defined the same way as $\emptyset$ except that all quantification is with special class variables.

By T.7, = is absolute, since $x \overset{s}{=} y \leftrightarrow x=y$. Also the union, intersection, complement, null class and universal class are absolute as they are defined using v, &, ~, 0 and 1, respectively. Also, by T.2 and T.3, I and Cl are absolute. If X is an individual, then $\{X\}$ is absolute. But, if X is a class, $\{X\}$ is not absolute, by similar reasoning to that of the previous chapter. Similarly, $\mathcal{P}(F)$ is not absolute.

The next task is to prove the consistency, relative to Z-F, of the above theory. The proof is similar to that of the last chapter but has to take account of individuals and non-significance. The domain of special classes and individuals can be taken as denumerable since, in Chapter 2, it was shown that a consistent applied predicate calculus with the 3-valued significance logic has a model with a denumerable domain. Let the domain be D^S and the model be N. D^S contains special class and individual constants, the value of the membership between any two of which is determined in N. To construct the model of the whole system, we need to extend the wffs of the 4-valued theory by adding the special class and individual constants of the model N, a, b, c,......., and some terms to be defined. The domain of the model will consist of some of these terms as well as the special class and individual constants. We give the formation rules for terms and wffs as follows :

1. If x and y are special class and individual variables, a and b are special class or individual constants, and X and Y are class and individual variables, then $a\epsilon b$, $a\epsilon x$, $x\epsilon a$, $a\epsilon X$, $X\epsilon a$, $x\epsilon y$, $x\epsilon X$, $X\epsilon x$, $X\epsilon Y$, aob, aox, xoa, aoX, Xoa, xoy, xoX, Xox, XoY, are all atomic wffs.

2. Any combination of wffs using $\sim$, &, $\supset$, T_n, A, S(quantifier), as in the 4-valued logic, is a wff.

3. A propositional constant (1, $\frac{1}{2}$, 0 or n) is an atomic wff.

Call a predicate A(X) or A(x) a <u>restricting predicate</u> if the following are satisfied :

(i) A(X) or A(x) is constructed from atomic wffs using only $\sim$, &, v, S(sig.), A, S(some), where the quantifiers A and S are unrestricted.

(ii) Each term and each variable over classes and individuals in A is covered by an 'S' (sig.).

(iii) A(X) or A(x) has the value 1 for some X or x, respectively, assuming the valuations for members of D^S, the significance of class-membership, the non-significance of individual-membership, the significance of the overlapping of individuals and the non-significance otherwise. These valuations and significance conditions are sufficient to evaluate A(X) or A(x) because of (i) and (ii).

4. A propositional constant or a wff constructed from atomic wffs

using only ~, &, v, S(sig.), A, S(some), where the quantifiers A and S can be unrestricted (using variables of type x or X) or restricted using a restricting predicate A(X) or A(x) as defined above, is a <u>standard wff</u>.

5. If P is a standard wff and is significant for all substitutions into its free variables (using the significance conditions of (iii)) and X is a class and individual variable, which does not occur free in predicates used to restrict variables in P, then $\{X : P\}$ is a term.

6. If $\{X : P\}$ and $\{X : Q\}$ are terms, y is a special class and individual variable, a is a special class or individual constant and Y is a class and individual variable, then $\{X : P\}\epsilon a$, $a\epsilon\{X : P\}$, $\{X : P\}\epsilon y$, $y\epsilon\{X : P\}$, $\{X : P\}\epsilon Y$, $Y\epsilon\{X : P\}$, $\{X : P\}\epsilon\{X : Q\}$, and each of these with '$\not\epsilon$' in place of 'ϵ', are all atomic wffs.

We construct a model for the axioms with domain the set D of all special class and individual constants and all constant terms $\{X : P\}$, i.e. P has no free variables at all or has X as its only free variable. Hence $D-D^S$ is the set of all constant terms. We shall use constants A, B, C,....., for members of D. Non-constant terms can be defined as follows : Associate with any term $\{X : P(X,z_1,\dots,z_m,Z_1,\dots,Z_n)\}$, for which $z_1,\dots,z_m,Z_1,\dots,Z_n$ are the only free variables, the function which for constants $a_1,\dots,a_m$ of D^S and $A_1,\dots,A_n$ of D takes as value the constant term $\{X : P(X,a_1,\dots,a_m,A_1,\dots,A_n)\}$ of D.

Let any specification of values including the value assignments already given to members of D^S in the model N and satisfying the significance conditions given in the formation rules, for all constant atomic wffs $A \in B$, where A and $B \in D$, be called a <u>structure on D</u>. Let $v_M(P)$ denote the value of the constant wff P given by the structure M on D. Also let $v_M(1)=1$, $v_M(\frac{1}{2})=\frac{1}{2}$, $v_M(0)=0$ and $v_M(n)=n$. Note that the values of the predicates $A(x)$ and $A(X)$, used to restrict variables, are either 1, 0 or n and are fixed for all structures. Define $M_1 \leq M_2$ for two structures M_1 and M_2 on D as, for any constant atomic wff P, if $v_{M_1}(P)=1$ then $v_{M_2}(P)=1$, if $v_{M_1}(P)=0$ then $v_{M_2}(P)=0$, and $v_{M_1}(P)=n$ iff $v_{M_2}(P)=n$. '$\leq$' defines a partial ordering on the set of structures, since (i) $M \leq M$, (ii) if $M_1 \leq M_2$ and $M_2 \leq M_3$ then $M_1 \leq M_3$, and (iii) if $M_1 \leq M_2$ and $M_2 \leq M_1$ then $M_1 = M_2$ (i.e. M_1 and M_2 are the same structure.).

<u>Lemma 1.</u>

Let M and M' be two structures on D, such that $M \leq M'$. Then, for any standard wff P, if $v_M(P)=1$ then $v_{M'}(P)=1$, if $v_M(P)=0$ then $v_{M'}(P)=0$, and $v_M(P)=n$ iff $v_{M'}(P)=n$.

<u>Proof.</u> By induction on wff evaluation procedure. This means that we start at the values of all the constant atomic wffs obtained by substitution for free variables in P, and then build up the value of P from these values according to the connectives and quantifiers.

If P is an atomic wff, the lemma holds.

(i) Let $v_M(\sim Q)=1$. Then $v_M(Q)=0$. By ind. hyp., $v_{M'}(Q)=0$. Hence $v_{M'}$

$(\sim Q)=1$. Let $v_M(\sim Q)=0$. Then $v_M(Q)=1$. By ind. hyp., $v_{M'}(Q)=1$ and $v_{M'}(\sim Q)=0$. Let $v_M(\sim Q)=n$. Then $v_M(Q)=n$. By ind. hyp., $v_{M'}(Q)=n$ and $v_{M'}(\sim Q)=n$. Let $v_{M'}(\sim Q)=n$. Then $v_{M'}(Q)=n$. By ind. hyp., $v_M(Q)=n$ and $v_M(\sim Q)=n$.

(ii) Let $v_M(Q \,\&\, R)=1$. Then $v_M(Q)=1$ and $v_M(R)=1$. By ind. hyp., $v_{M'}(Q)=1$ and $v_{M'}(R)=1$. Hence $v_{M'}(Q \,\&\, R)=1$.

Let $v_M(Q \,\&\, R)=0$. Then $v_M(Q)=0$ or $v_M(R)=0$, $v_M(Q)\neq n$ and $v_M(R)\neq n$. By ind. hyp., $v_{M'}(Q)=0$ or $v_{M'}(R)=0$, $v_{M'}(Q)\neq n$ and $v_{M'}(R)\neq n$. Hence $v_{M'}(Q \,\&\, R)=0$.

Let $v_M(Q \,\&\, R)=n$. Then $v_M(Q)=n$ or $v_M(R)=n$. By ind. hyp., $v_{M'}(Q)=n$ or $v_{M'}(R)=n$. Hence $v_{M'}(Q \,\&\, R)=n$.

Let $v_{M'}(Q \,\&\, R)=n$. Then $v_{M'}(Q)=n$ or $v_{M'}(R)=n$. By ind. hyp., $v_M(Q)=n$ or $v_M(R)=n$. Hence $v_M(Q \,\&\, R)=n$.

(iii) Let $v_M(Q \vee R)=1$. Hence $v_M(Q)=1$ or $v_M(R)=1$. By ind. hyp., $v_{M'}(Q)=1$ or $v_{M'}(R)=1$. Hence $v_{M'}(Q \vee R)=1$.

Let $v_M(Q \vee R)=0$. Then $v_M(R)=0$ and $v_M(Q)=0$ or n, or $v_M(R)=n$ and $v_M(Q)=0$. By ind. hyp., $v_{M'}(R)=0$ and $v_{M'}(Q)=0$ or n, or $v_{M'}(R)=n$ and $v_{M'}(Q)=0$. Hence $v_{M'}(Q \vee R)=0$.

Let $v_M(Q \vee R)=n$. Then $v_M(Q)=n$ and $v_M(R)=n$. By ind. hyp., $v_{M'}(Q)=n$ and $v_{M'}(R)=n$. Hence $v_{M'}(Q \vee R)=n$.

Let $v_{M'}(Q \vee R)=n$. Then $v_{M'}(Q)=n$ and $v_{M'}(R)=n$. By ind. hyp., $v_M(Q)=n$ and $v_M(R)=n$ and hence $v_M(Q \vee R)=n$.

(iv) Let $v_M(SQ)=1$. Then $v_M(Q)\neq n$. By ind. hyp., $v_{M'}(Q)\neq n$ and hence $v_{M'}(SQ)=1$.

Let $v_M(SQ)=0$. Then $v_M(Q)=n$. By ind. hyp., $v_{M'}(Q)=n$ and hence $v_{M'}(SQ)=0$.

SQ cannot take the value n.

(v) Let $v_M((Ax)Q(x))=1$. Then $v_M(Q(x))=1$ for all $x \in D^S$. By ind. hyp., $v_{M'}(Q(x))=1$, for all $x \in D^S$, and hence $v_{M'}((Ax)Q(x))=1$.

Let $v_M((Ax)Q(x))=0$. Then $v_M(Q(x)) \neq n$ for all $x \in D^S$ and $v_M(Q(x))=0$ for some $x \in D^S$. By ind. hyp., $v_{M'}(Q(x)) \neq n$ for all $x \in D^S$ and $v_{M'}(Q(x))=0$ for some $x \in D^S$. Hence $v_{M'}((Ax)Q(x))=0$.

Let $v_M((Ax)Q(x))=n$. Then $v_M(Q(x))=n$ for some $x \in D^S$. By ind. hyp., $v_{M'}(Q(x))=n$ for some $x \in D^S$ and hence $v_{M'}((Ax)Q(x))=n$.

Let $v_{M'}((Ax)Q(x))=n$. Then $v_{M'}(Q(x))=n$ for some $x \in D^S$. By ind. hyp., $v_M(Q(x))=n$ for some $x \in D^S$ and hence $v_M((Ax)A(x))=n$.

(vi) The case for $(AX)Q(X)$ is similar to (v).

(vii) Let $v_M((Sx)Q(x))=1$. Then $v_M(Q(x))=1$ for some $x \in D^S$. By ind. hyp., $v_{M'}(Q(x))=1$ for some $x \in D^S$ and hence $v_{M'}((Sx)Q(x))=1$.

Let $v_M((Sx)Q(x))=0$. Then $v_M(Q(x))=0$ or n for all $x \in D^S$ and $v_M(Q(x)) \neq n$ for some $x \in D^S$. By ind. hyp., $v_{M'}(Q(x))=0$ or n for all $x \in D^S$ and $v_{M'}(Q(x)) \neq n$ for some $x \in D^S$. Hence $v_{M'}((Sx)Q(x))=0$.

Let $v_M((Sx)Q(x))=n$. Then $v_M(Q(x))=n$ for all $x \in D^S$. By ind. hyp., $v_{M'}(Q(x))=n$ for all $x \in D^S$ and hence $v_{M'}((Sx)Q(x))=n$.

Let $v_{M'}((Sx)Q(x))=n$. Then $v_{M'}(Q(x))=n$ for all $x \in D^S$. By ind. hyp., $v_M(Q(x))=n$ for all $x \in D^S$ and hence $v_M((Sx)Q(x))=n$.

(viii) The case for $(SX)Q(X)$ is similar to (vii).

(ix) Let $v_M((Ax)(A(x) \supset Q(x)))=1$. Then $v_M(A(x) \supset Q(x))=1$ for all

$x \notin D^S$, and hence $v_M(A(x))=0$ or n or, $v_M(A(x))=1$ and $v_M(Q(x))=1$, for all $x \notin D^S$. By ind. hyp. and the conditions on $A(x)$, $v_{M'}(A(x))=0$ or n or, $v_{M'}(A(x))=1$ and $v_{M'}(Q(x))=1$, for all $x \notin D^S$. Hence $v_{M'}((Ax)(A(x) \supset Q(x)))=1$.

Let $v_M((Ax)(A(x) \supset Q(x)))=0$. Then $v_M(A(x) \supset Q(x))=0$ for some $x \notin D^S$ and $v_M(A(x) \supset Q(x)) \neq n$ for all $x \notin D^S$. Hence $v_M(A(x))=1$ and $v_M(Q(x))=0$ for some $x \notin D^S$, and $v_M(A(x))=0$ or n or, $v_M(A(x))=1$ and $v_M(Q(x)) \neq n$, for all $x \notin D^S$. By ind. hyp. and the conditions on $A(x)$, the same holds for M' and hence $v_{M'}((Ax)(A(x) \supset Q(x)))=0$.

Let $v_M((Ax)(A(x) \supset Q(x)))$

(see next page.)

$=n$. Then $v_M(A(x) \supset Q(x))=n$ for some $x \in D^S$, and $v_M(A(x))=1$ and $v_M(Q(x))$ $=n$ for some $x \in D^S$. By ind. hyp. and the conditions on $A(x)$, $v_{M'}(A(x))$ $=1$ and $v_{M'}(Q(x))=n$, and so $v_{M'}((Ax)(A(x) \supset Q(x)))=n$. Let $v_{M'}((Ax)$ $(A(x) \supset Q(x)))=n$. Then $v_{M'}(A(x) \supset Q(x))=n$ for some $x \in D^S$ and hence $v_{M'}(A(x))=1$ and $v_{M'}(Q(x))=n$ for some $x \in D^S$. By ind. hyp. and the conditions on $A(x)$, $v_M(A(x))=1$ and $v_M(Q(x))=n$ for some $x \in D^S$ and hence $v_M((Ax)(A(x) \supset Q(x)))=n$.

(x) The case for $(AX)(A(X) \supset Q(X))$ is similar to (ix).

(xi) Let $v_M((Sx)(T_nA(x) \& Q(x)))=1$. Then $v_M(T_nA(x) \& Q(x))=1$ for some $x \in D^S$, and $v_M(A(x))=1$ and $v_M(Q(x))=1$ for some $x \in D^S$. By ind. hyp. and the conditions on $A(x)$, $v_{M'}(A(x))=1$ and $v_{M'}(Q(x))=1$ for some $x \in D^S$. Hence $v_{M'}((Sx)(T_nA(x) \& Q(x)))=1$.

Let $v_M((Sx)(T_nA(x) \& Q(x)))=0$. Then $v_M(T_nA(x) \& Q(x))=0$ or n for all $x \in D^S$ and $v_M(T_nA(x) \& Q(x)) \neq n$ for some $x \in D^S$. Then $v_M(A(x))=0$ or n, or $v_M(Q(x))=0$ cr n for all $x \in D^S$, and $v_M(A(x))=1$ and $v_M(Q(x)) \neq n$ for some $x \in D^S$ By ind. hyp. and the conditions on $A(x)$, the same applies with M' for M. Hence $v_{M'}((Sx)(T_nA(x) \& Q(x)))=0$.

Let $v_M((Sx)(T_nA(x) \& Q(x)))=n$. Then $v_M(T_nA(x) \& Q(x))=n$ for all $x \in D^S$. Hence $v_M(A(x))=0$ or n or $v_M(Q(x))=n$ for all $x \in D^S$. By ind. hyp. and the conditions on $A(x)$, $v_{M'}(A(x))=0$ or n or $v_{M'}(Q(x))=n$ for all $x \in D^S$. Hence $v_{M'}((Sx)(T_nA(x) \& Q(x)))=n$. Let $v_{M'}((Sx)(T_nA(x) \& Q(x)))=n$. Then $v_{M'}(A(x))=0$ or n or $v_{M'}(Q(x))=n$ for all $x \in D^S$. By ind. hyp. and the conditions on $A(x)$, $v_M(A(x))=0$ or n or $v_M(Q(x))=n$ for all $x \in D^S$ and hence $v_M((Sx)(T_nA(x) \& Q(x)))=n$.

(xii) The case for $(SX)(T_n A(X)\ \&\ Q(X))$ is similar to (xi).

Define the structure M_0 as follows:

If $A \notin D^S$ or $B \notin D^S$, then $v_{M_0}(A \in B) = \frac{1}{2}$, where B is a class. [By the definition of structures in general, $v_{M_0}(A \in B) = n$, where B is an individual.]

If $A \in D^S$ and $B \in D^S$, then $v_{M_0}(A \in B) = 1$ if $A \in B$ is true in the model N, $=0$ if $A \in B$ is false in the model N, and $=n$ if $A \in B$ is non-significant in the model N.

If A is a class or B is a class then $v_{M_0}(AoB) = n$.

If A and B are individuals then $v_{M_0}(AoB) =$ the value given in the model N.

Hence M_0 with domain D^S is a model for all the special class and individual axioms. The model of the whole system will be the limit of a sequence of structures, $M_0 \leq M_1 \leq \ldots \leq M_\mu \ldots$, on D.

Assuming M_μ defined for some ordinal μ, $M_{\mu+1}$ is defined as follows:

$v_{M_{\mu+1}}(A \in \{X : P(X)\}) = v_{M_\mu}(P(A))$.

If b is a individual, then $v_{M_{\mu+1}}(A \in b) = n$.

Now let b be a special class and $A \in D - D^S$. If $\sim z \in A \vee z \in a . \& . \sim z \in a \vee z \in A$ is valid in M_μ for some special class a, then, for all special classes b, $v_{M_{\mu+1}}(A \in b) = v_{M_0}(a \in b)$. If $(Af)(Sz)(z \in A\ \&\ \sim z \in f . v . z \in f\ \&\ \sim z \in A)$ has the value 1 in M_μ then, for all special classes b, $v_{M_{\mu+1}}(A \in b) = 0$. If neither $(Sf)(Az)(\sim z \in A \vee z \in f . \& . \sim z \in f \vee z \in A)$ nor $(Af)(Sz)(z \in A\ \&\ \sim z \in f . v . z \in f\ \&\ \sim z \in A)$ has the value 1 in M_μ then $v_{M_{\mu+1}}(A \in b) = \frac{1}{2}$, for all special classes b.

If A is a class or B is a class then $v_{M_{\mu+1}}(AoB)=n$. If A and B are individuals then $v_{M_{\mu+1}}(AoB)=$ the value given in the model N.

For a limit ordinal μ, on the assumption that $M_\nu \leq M_\tau$ for all $\nu \leq \tau$, for all $\tau < \mu$, for all atomic wffs P, if $v_{M_\nu}(P)=1$ for some $\nu < \mu$ then $v_{M_\mu}(P)=1$, if $v_{M_\nu}(P)=0$ for some $\nu < \mu$ then $v_{M_\mu}(P)=0$, if $v_{M_\nu}(P)=\frac{1}{2}$ for all $\nu < \mu$ then $v_{M_\mu}(P)=\frac{1}{2}$, and if $v_{M_\nu}(P)=n$ for some $\nu < \mu$ then $v_{M_\mu}(P)=n$.

Lemma 2.

$M_\nu \leq M_\mu$, for all $\nu \leq \mu$.

Proof. By transfinite induction on μ. The induction hypothesis is : $M_\nu \leq M_\tau$ for all $\nu \leq \tau$, for all $\tau < \mu$.

It is clear that $v_{M_\nu}(AoB)=v_{M_\mu}(AoB)$, for all A and B.

(i) $\mu=0$. $M_0 \leq M_0$.

(ii) μ is a successor ordinal.

(A) Let $v_{M_\nu}(A\in\{X : P\})=1$. There is a $\eta < \nu$ such that $v_{M_\eta}(P(A))=1$ by the method of construction of the structures. Since $\eta \leq \mu-1$, $M_\eta \leq M_{\mu-1}$ by the ind. hyp. Hence $v_{M_{\mu-1}}(P(A))=1$. By the construction of M_μ, $v_{M_\mu}(A\in\{X : P\})=1$. Similarly, if $v_{M_\nu}(A\in\{X : P\})=0$ then $v_{M_\mu}(A\in\{X : P\})$ $=0$.

(B) Let $v_{M_\nu}(A\in b)=n$, b being an individual. Then $v_{M_\mu}(A\in b)=n$, independently of $v_{M_\nu}(A\in b)=n$. Similarly, vice versa.

(C) Let $v_{M_\nu}(A\in b)=1$ (or 0), b being a special class. There is an $\eta < \nu$ such that $v_{M_\eta}((Sf)(Az)(\sim z\in A \vee z\in f.\&.\sim z\in f \vee z\in A))=1$ or $v_{M_\eta}((Af)(Sz)$ $(z\in A \;\&\; \sim z\in f.\vee.z\in f \;\&\; \sim z\in A))=1$.

(a) Let $v_{M_\eta}((Sf)(Az)(\sim z\epsilon A \vee z\epsilon f.\&.\sim z\epsilon f \vee z\epsilon A))=1$. Then $v_{M_{\eta+1}}(A\epsilon b)$ $=v_{M_0}(a\epsilon b)=1$ (or 0), for some special class a. Since $\eta\leq\mu-1$, $M_\eta\leq M_{\mu-1}$, by the ind. hyp. Hence $v_{M_{\mu-1}}((Sf)(Az)(\sim z\epsilon A \vee z\epsilon f.\&.\sim z\epsilon f \vee z\epsilon A))=1$ and $v_{M_\mu}(A\epsilon b)=v_{M_0}(a\epsilon b)=1$ (or 0), for some special class a.

(b) Let $v_{M_\eta}((Af)(Sz)(z\epsilon A \;\&\; \sim z\epsilon f.\vee.z\epsilon f \;\&\; \sim z\epsilon A))=1$. If $v_{M_\nu}(A\epsilon b)=1$, this does not apply. Let $v_{M_\nu}(A\epsilon b)=0$. Since $\eta\leq\mu-1$, $M_\eta\leq M_{\mu-1}$, by the ind. hyp. Hence $v_{M_{\mu-1}}((Af)(Sz)(z\epsilon A \;\&\; \sim z\epsilon f.\vee.z\epsilon f \;\&\; \sim z\epsilon A))=1$ and $v_{M_\mu}(A\epsilon b)=0$.

(iii) <u>μ is a limit ordinal.</u>

Let $\nu<\mu$. Let $v_{M_\nu}(A\epsilon B)=1$. Then $v_{M_\mu}(A\epsilon B)=1$ by definition of M_μ. Similarly if $v_{M_\nu}(A\epsilon B)=0$ then $v_{M_\mu}(A\epsilon B)=0$, and if $v_{M_\nu}(A\epsilon B)=n$ then $v_{M_\mu}(A\epsilon B)=n$. Also if $v_{M_\mu}(A\epsilon B)=n$ then B is an individual and $v_{M_\nu}(A\epsilon B)$ $=n$. If $\nu=\mu$, $M_\nu\leq M_\mu$.

<u>Lemma 3.</u>

There is an ordinal λ of the second number class such that $M_\lambda=M_{\lambda+1}$.

<u>Proof.</u> The increasing chain of structures, $M_0\leq M_1\leq\ldots\leq M_\mu\leq\ldots$, can be regarded as two increasing chains of subsets of the denumerable set of all atomic wffs of the form $A\epsilon B$, B being a class. One chain is of those atomic wffs taking the value 1 and the other is of those taking the value 0. If $M_\nu=M_{\nu+1}$ then $M_\nu=M_\mu$ for all ordinals μ, $\nu\leq\mu$, since, by the method of construction, there is no way of changing the values of any atomic wffs. There is a denumerable set of ordinals μ such that $M_\mu\neq M_{\mu+1}$. But the set of all ordinals of the second number

class is non-denumerable, and hence for some λ in this class, $M_\lambda = M_{\lambda+1}$.

Now it is required to show that M_λ is the required model.

Theorem 1.

All Individual Axioms, all Special Class Axioms and General Axioms 6 and 7 are valid in M_λ.

Proof. By the definitions of M_0 and the domain D^S, all the above axioms are valid in M_0 with D^S as domain. By lemma 2, if $v_{M_0}(A \epsilon B) = 1$, 0 or n then $v_{M_\lambda}(A \epsilon B) = 1$, 0 or n, resp. Also $v_{M_0}(AoB) = v_{M_\lambda}(AoB)$.

Theorem 2.

$Y \epsilon \{X : P\} \leftrightarrow P(Y)$ is valid in M_λ.

Proof. Let $v_{M_\lambda}(A \epsilon \{X : P\}) = 1$. Let ν be the least ordinal such that $v_{M_\nu}(A \epsilon \{X : P\}) = 1$. ν is a successor ordinal. Hence $v_{M_{\nu-1}}(P(A)) = 1$. Since $\nu - 1 \leq \lambda$, $M_{\nu-1} \leq M_\lambda$, by lemma 2. Since P is a standard wff, by lemma 1, $v_{M_\lambda}(P(A)) = 1$. Similarly, if $v_{M_\lambda}(A \epsilon \{X : P\}) = 0$, then $v_{M_\lambda}(P(A)) = 0$.

Let $v_{M_\lambda}(P(A)) = 1$. Then $v_{M_{\lambda+1}}(A \epsilon \{X : P\}) = 1$. Since $M_\lambda = M_{\lambda+1}$, $v_{M_\lambda}(A \epsilon \{X : P\}) = 1$. Similarly, if $v_{M_\lambda}(P(A)) = 0$, then $v_{M_\lambda}(A \epsilon \{X : P\}) = 0$.

Theorem 3.

The Abstraction Axiom (A) is valid in M_λ.

Proof. By Theorem 2, for any standard wff P which is significant for all substitutions into its free variables, $Y \epsilon \{X : P\} \leftrightarrow P(Y)$ is valid in M_λ. Therefore $(AX, y_1, \ldots, y_m, Y_1, \ldots, Y_n) S\phi(X, y_1, \ldots, y_m, Y_1, \ldots, Y_n) \supset (SF)(AX)(X \epsilon F \leftrightarrow \phi(X, y_1, \ldots, y_m, Y_1, \ldots, Y_n))$ is valid in M_λ, for all wffs $\emptyset$ of the required sort.

Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. Let ν_P be the least ordinal such that $v_{M_{\nu_P}}(P)=1$ or 0. Form the set of all constant atomic wffs of P (i.e. atomic wffs of P with all substitutions made for any variables that occur in them) which do not occur in a predicate used to restrict variables, which are not covered by an 'S' (sig.), and which take the value 1 or 0 in M_{ν_P}. Call this <u>the dependent set of P, D(P)</u>.

<u>Lemma 4</u>.

Let A and B be classes. Let P(A) be a standard wff such that A does not occur in any predicate used to restrict variables. Then $v_{M_\lambda}(SP(A))=v_{M_\lambda}(SP(B))$.

<u>Proof</u>. By induction on wff evaluation procedure. It is clear in the case of an atomic wff.

(i) Let P(A) be ~R(A). By ind. hyp., $v_{M_\lambda}(SR(A))=v_{M_\lambda}(SR(B))$ and hence $v_{M_\lambda}(S{\sim}R(A))=v_{M_\lambda}(S{\sim}R(B))$.

(ii) Let P(A) be R(A) & S(A). By ind. hyp., $v_{M_\lambda}(SR(A))=v_{M_\lambda}(SR(B))$ and $v_{M_\lambda}(SS(A))=v_{M_\lambda}(SS(B))$. $v_{M_\lambda}(S(R(A)\ \&\ S(A)))=v_{M_\lambda}(SR(A)\ \&\ SS(A))=v_{M_\lambda}(SR(B)\ \&\ SS(B))=v_{M_\lambda}(S(R(B)\ \&\ S(B)))$.

(iii) Let P(A) be R(A) v S(A). By ind. hyp., $v_{M_\lambda}(SR(A))=v_{M_\lambda}(SR(B))$ and $v_{M_\lambda}(SS(A))=v_{M_\lambda}(SS(B))$. $v_{M_\lambda}(S(R(A)\ v\ S(A)))=v_{M_\lambda}(SR(A)\ v\ SS(A))=v_{M_\lambda}(SR(B)\ v\ SS(B))=v_{M_\lambda}(S(R(B)\ v\ S(B)))$.

(iv) Let $P(A)$ be $SR(A)$. By ind. hyp., $v_{M_\lambda}(SR(A))=v_{M_\lambda}(SR(B))$. Hence $v_{M_\lambda}(SSR(A))=v_{M_\lambda}(SSR(B))$.

(v) Let $P(A)$ be $(Ax)R(A,x)$. By ind. hyp., $v_{M_\lambda}(SR(A,x))=v_{M_\lambda}(SR(B,x))$, for all $x\in D^S$. $v_{M_\lambda}(S(Ax)R(A,x))=v_{M_\lambda}((Ax)SR(A,x))=v_{M_\lambda}((Ax)SR(B,x))=v_{M_\lambda}(S(Ax)R(B,x))$.

(vi) Similarly, if $P(A)$ is $(AX)R(A,X)$.

(vii) Let $P(A)$ be $(Sx)R(A,x)$. By ind. hyp., $v_{M_\lambda}(SR(A,x))=v_{M_\lambda}(SR(B,x))$ for all $x\in D^S$. $v_{M_\lambda}(S(Sx)R(A,x))=v_{M_\lambda}((Sx)SR(A,x))=v_{M_\lambda}((Sx)SR(B,x))=v_{M_\lambda}(S(Sx)R(B,x))$.

(viii) Similarly if $P(A)$ is $(SX)R(A,X)$.

(ix) Let $P(A)$ be $(Ax)(A(x) \supset R(A,x))$. By ind. hyp., $v_{M_\lambda}(SR(A,x))=v_{M_\lambda}(SR(B,x))$ for all $x\in D^S$. $v_{M_\lambda}(S(Ax)(A(x) \supset R(A,x)))=v_{M_\lambda}((Ax)(A(x) \supset SR(A,x)))=v_{M_\lambda}((Ax)(A(x) \supset SR(B,x)))=v_{M_\lambda}(S(Ax)(A(x) \supset R(B,x)))$.

(x) Similarly if $P(A)$ is $(AX)(A(X) \supset R(A,X))$.

(xi) Let $P(A)$ be $(Sx)(T_nA(x) \,\&\, R(A,x))$. By ind. hyp., $v_{M_\lambda}(SR(A,x))=v_{M_\lambda}(SR(B,x))$ for all $x\in D^S$. $v_{M_\lambda}(S(Sx)(T_nA(x) \,\&\, R(A,x)))=v_{M_\lambda}((Sx)(T_nA(x) \,\&\, SR(A,x)))=v_{M_\lambda}((Sx)(T_nA(x) \,\&\, SR(B,x)))=v_{M_\lambda}(S(Sx)(T_nA(x) \,\&\, R(B,x)))$.

(xii) Similarly if $P(A)$ is $(SX)(T_nA(X) \,\&\, R(A,X))$.

Lemma 5.

Let A and B be classes. Let $P(A)$ be a standard wff such that A does not occur in any predicate used to restrict variables and such that $v_{M_\lambda}(P(A))=1$ or 0. If, for each $Q(A)\in D(P(A))$ $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$, then $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

Proof. By induction on wff evaluation procedure. If $P(A)$ is an atomic wff such that $v_{M_\lambda}(P(A))=1$ or 0, then $D(P(A))=\{P(A)\}$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(i) Let $P(A)$ be $\sim R(A)$. Since $D(\sim R(A))=D(R(A))$, for each $Q(A)\in D(R(A))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(ii) Let $P(A)$ be $R(A)\ \&\ S(A)$ and $v_{M_\lambda}(R(A)\ \&\ S(A))=1$. Then $v_{M_\lambda}(R(A))=1$ and $v_{M_\lambda}(S(A))=1$. Since $\nu_{R(A)}\leq\nu_{R(A)\ \&\ S(A)}$, $D(R(A))\subseteq D(R(A)\ \&\ S(A))$. Hence, for each $Q(A)\in D(R(A))$, $v_{M_\lambda}(Q(B))=v_{M_\lambda}(Q(A))$. By ind. hyp., $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))$. Similarly, $v_{M_\lambda}(S(B))=v_{M_\lambda}(S(A))$. Hence $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iii) Let $P(A)$ be $R(A)\ \&\ S(A)$ and $v_{M_\lambda}(R(A)\ \&\ S(A))=0$. Then $v_{M_\lambda}(R(A))=0$ or $v_{M_\lambda}(S(A))=0$, and $v_{M_\lambda}(R(A))\neq n$ and $v_{M_\lambda}(S(A))\neq n$. Since $\nu_{R(A)}=\nu_{R(A)\ \&\ S(A)}$ or $\nu_{S(A)}=\nu_{R(A)\ \&\ S(A)}$, $D(R(A))\subseteq D(R(A)\ \&\ S(A))$ or $D(S(A))\subseteq D(R(A)\ \&\ S(A))$. Hence, as above, $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))=0$ or $v_{M_\lambda}(S(A))=v_{M_\lambda}(S(B))=0$. By lemma 4, $v_{M_\lambda}(R(B)\ \&\ S(B))=0$ and $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(iv) Let $P(A)$ be $R(A)\ v\ S(A)$ and $v_{M_\lambda}(R(A)\ v\ S(A))=1$. Then $v_{M_\lambda}(R(A))=1$ or $v_{M_\lambda}(S(A))=1$. Since $\nu_{R(A)}=\nu_{R(A)\ v\ S(A)}$ or $\nu_{S(A)}=\nu_{R(A)\ v\ S(A)}$, $D(R(A))\subseteq D(R(A)\ v\ S(A))$ or $D(S(A))\subseteq D(R(A)\ v\ S(A))$. Hence, as above, $v_{M_\lambda}(R(B))=v_{M_\lambda}(R(A))=1$ or $v_{M_\lambda}(S(B))=v_{M_\lambda}(S(A))=1$. Hence $v_{M_\lambda}(R(B)\ v\ S(B))=1$.

(v) Let $P(A)$ be $R(A)\ v\ S(A)$ and $v_{M_\lambda}(R(A)\ v\ S(A))=0$. Then (I) $v_{M_\lambda}(R(A))=0$ and $v_{M_\lambda}(S(A))=0$, (II) $v_{M_\lambda}(R(A))=0$ and $v_{M_\lambda}(S(A))=n$, or (III) $v_{M_\lambda}(R(A))=n$ and $v_{M_\lambda}(S(A))=0$.

In case (I), $\nu_{R(A)} \leq \nu_{R(A) \vee S(A)}$, $D(R(A)) \subseteq D(R(A) \vee S(A))$, and hence $v_{M_\lambda}(R(B)) = v_{M_\lambda}(R(A))$. Similarly, $v_{M_\lambda}(S(B)) = v_{M_\lambda}(S(A))$. Hence $v_{M_\lambda}(R(B) \vee S(B)) = v_{M_\lambda}(R(A) \vee S(A))$.

In case (II), $\nu_{R(A)} = \nu_{R(A) \vee S(A)}$, $D(R(A)) \subseteq D(R(A) \vee S(A))$, and hence $v_{M_\lambda}(R(B)) = v_{M_\lambda}(R(A))$. By lemma 4, $v_{M_\lambda}(S(B)) = v_{M_\lambda}(S(A))$ and hence $v_{M_\lambda}(R(A) \vee S(A)) = v_{M_\lambda}(R(B) \vee S(B))$.

Case (III) is similar.

(vi) Let $P(A)$ be $SR(A)$. By lemma 4, $v_{M_\lambda}(SR(A)) = v_{M_\lambda}(SR(B))$.

(vii) Let $P(A)$ be $(Ax)R(A,x)$ and let $v_{M_\lambda}(P(A)) = 1$. Then $v_{M_\lambda}(R(A,x)) = 1$ for all $x \in D^S$. $\nu_{R(A,x)} \leq \nu_{(Ax)R(A,x)}$ and $D(R(A,x)) \subseteq D((Ax)R(A,x))$ for all $x \in D^S$. Hence, $v_{M_\lambda}(R(A,x)) = v_{M_\lambda}(R(B,x))$ for all $x \in D^S$, and $v_{M_\lambda}((Ax)R(A,x)) = v_{M_\lambda}((Ax)R(B,x))$.

(viii) Let $P(A)$ be $(Ax)R(A,x)$ and let $v_{M_\lambda}(P(A)) = 0$. Then $v_{M_\lambda}(R(A,x)) = 0$ for some $x \in D^S$ and $v_{M_\lambda}(R(A,x)) \neq n$ for all $x \in D^S$. $\nu_{R(A,x)} = \nu_{(Ax)R(A,x)}$ for some $x \in D^S$. Hence $D(R(A,x)) \subseteq D(P(A))$ and $v_{M_\lambda}(R(A,x)) = v_{M_\lambda}(R(B,x)) = 0$ for this x. Hence $v_{M_\lambda}((Ax)R(A,x)) = v_{M_\lambda}((Ax)R(B,x))$, by using lemma 4.

(ix) The case for $P(A)$ being $(AX)R(A,X)$ is similar to (vii) and (viii).

(x) Let $P(A)$ be $(Sx)R(A,x)$ and let $v_{M_\lambda}(P(A)) = 1$. Then $v_{M_\lambda}(R(A,x)) = 1$ for some $x \in D^S$. Since $\nu_{R(A,x)} = \nu_{(Sx)R(A,x)}$ and $D(R(A,x)) \subseteq D((Sx)R(A,x))$ for some $x \in D^S$, $v_{M_\lambda}(R(A,x)) = v_{M_\lambda}(R(B,x))$ for this x. Hence $v_{M_\lambda}((Sx)R(A,x)) = v_{M_\lambda}((Sx)R(B,x))$.

(xi) Let $P(A)$ be $(Sx)R(A,x)$ and let $v_{M_\lambda}(P(A)) = 0$. Then $v_{M_\lambda}(R(A,x)) = 0$ or n for all $x \in D^S$, and $v_{M_\lambda}(R(A,x)) = 0$ for some $x \in D^S$. If $v_{M_\lambda}(R(A,x)) = n$ then $v_{M_\lambda}(R(B,x)) = n$, by lemma 4. If $v_{M_\lambda}(R(A,x)) = 0$ then $\nu_{R(A,x)} \leq$

$\nu_{P(A)}$ and $D(R(A,x)) \subseteq D(P(A))$. Hence $v_{M_\lambda}(R(A,x))=v_{M_\lambda}(R(B,x))$. Hence $v_{M_\lambda}((Sx)R(A,x))=v_{M_\lambda}((Sx)R(B,x))$.

(xii) The case for $P(A)$ being $(SX)R(A,X)$ is similar to (x) and (xi).

(xiii) Let $P(A)$ be $(Ax)(A(x) \supset R(A,x))$ and let $v_{M_\lambda}(P(A))=1$. Then $v_{M_\lambda}(A(x) \supset R(A,x))=1$, for all $x \in D^S$, and hence $v_{M_\lambda}(A(x))=0$ or n or, $v_{M_\lambda}(A(x))=1$ and $v_{M_\lambda}(R(A,x))=1$, for all $x \in D^S$. If $v_{M_\lambda}(R(A,x))=1$ and $v_{M_\lambda}(A(x))=1$ then $\nu_{R(A,x)} \leq \nu_{P(A)}$ and $D(R(A,x)) \subseteq D(P(A))$. Hence $v_{M_\lambda}(R(A,x))=v_{M_\lambda}(R(B,x))$, and $v_{M_\lambda}(P(A))=v_{M_\lambda}(P(B))$.

(xiv) Let $P(A)$ be $(Ax)(A(x) \supset R(A,x))$ and let $v_{M_\lambda}(P(A))=0$. Then $v_{M_\lambda}(A(x) \supset R(A,x))=0$ for some $x \in D^S$ and $v_{M_\lambda}(A(x) \supset R(A,x)) \neq n$ for all $x \in D^S$. If $v_{M_\lambda}(A(x) \supset R(A,x))=0$ for some $x \in D^S$ then $v_{M_\lambda}(A(x))=1$ and $v_{M_\lambda}(R(A,x))=0$ for that x. $\nu_{R(A,x)}=\nu_{P(A)}$ and $D(R(A,x)) \subseteq D(P(A))$ for some $x \in D^S$ such that $v_{M_\lambda}(A(x))=1$. Hence $v_{M_\lambda}(R(A,x))=v_{M_\lambda}(R(B,x))$ and, using lemma 4, $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(xv) If $P(A)$ is $(AX)(A(X) \supset R(A,X))$ then this case is similar to (xiii) and (xiv).

(xvi) Let $P(A)$ be $(Sx)(T_nA(x) \;\&\; R(A,x))$ and let $v_{M_\lambda}(P(A))=1$. Then $v_{M_\lambda}(T_nA(x) \;\&\; R(A,x))=1$ for some $x \in D^S$. Hence $v_{M_\lambda}(A(x))=1$ and $v_{M_\lambda}(R(A,x))=1$ for some $x \in D^S$. $\nu_{R(A,x)}=\nu_{P(A)}$ and $D(R(A,x)) \subseteq D(P(A))$ for some $x \in D^S$ such that $v_{M_\lambda}(A(x))=1$. Hence $v_{M_\lambda}(R(A,x))=v_{M_\lambda}(R(B,x))$ and $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

(xvii) Let $P(A)$ be $(Sx)(T_nA(x) \;\&\; R(A,x))$ and let $v_{M_\lambda}(P(A))=0$. Then $v_{M_\lambda}(T_nA(x) \;\&\; R(A,x))=0$ or n for all $x \in D^S$, and $v_{M_\lambda}(T_nA(x) \;\&\; R(A,x))=0$ for some $x \in D^S$. Hence, for all $x \in D^S$, $v_{M_\lambda}(A(x))=1$ and $v_{M_\lambda}(R(A,x))$

$=0$ or n. $v_{M_\lambda}(R(A,x))=0$ for some $x\notin D^S$. If $v_{M_\lambda}(R(A,x))=0$ then $\nu_{R(A,x)} \leq \nu_{P(A)}$ and $D(R(A,x)) \subseteq D(P(A))$. Hence $v_{M_\lambda}(R(A,x))=v_{M_\lambda}(R(B,x))$ and, using lemma 4, $v_{M_\lambda}(P(A))=v_{M_\lambda}(P(B))$.

(xviii) If $P(A)$ is $(SX)(T_n A(X)$ & $R(A,X))$ then this case is similar to (xvi) and (xvii).

Let P be an atomic wff of the form $A\epsilon\{X : Q(X)\}$ such that $v_{M_\lambda}(P)=1$ or 0. Define the corresponding standard wff of P, C_P, as $Q(A)$.

Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. Let P have dependent set, $D(P)$. We define a general dependent set of P as follows :

(i) The dependent set $D(P)$ of P is a gen. dep. set of P.

(ii) If $v_{M_\lambda}(R)=1$ or 0, R is an atomic wff (not 1 or 0) and C_R is defined for R, then $D(C_R)$ is a gen. dep. set of R.

(iii) Let D' be a gen. dep. set of P. Let $S \subseteq D'$. If $Q\in S$, let D_Q be a gen. dep. set of Q. Then $(D' \cap \bar{S}) \cup \bigcup_{Q\in S} D_Q$ is a gen. dep. set of P. This assumes $v_{M_\lambda}(Q)=1$ or 0, for all $Q\in S$. Note that lemma 6 should be coupled with the definition of a gen. dep. set so that the assumption can be made before the construction of the gen. dep. sets of D_Q.

Lemma 6.

Let P be a standard wff such that $v_{M_\lambda}(P)=1$ or 0. If D' is a gen. dep. set of P then, for each $Q\in D'$, $v_{M_\lambda}(Q)=1$ or 0.

Proof. The proof is the same as that for lemma 5 of the previous chapter.

Lemma 7.

Let P be an atomic wff such that $v_{M_\lambda}(P)=1$ or 0 and such that C_P is defined. If D' is a gen. dep. set of P which is not D(P) then, for each $Q \in D'$, $v_{M_{\mu_P-1}}(Q)=1$ or 0.

Proof. The proof is the same as that for lemma 6 of the previous chapter.

Lemma 8.

Let A and B be classes. Let P(A) be a standard wff such that A does not occur in any predicate used to restrict variables and such that $v_{M_\lambda}(P(A))=1$ or 0. Consider any gen. dep. set of P(A), such that, in the process of construction, (ii) is not applied to any atomic wff of the form $C \notin A$. If, for all $Q(A) \in D'$, $v_{M_\lambda}(Q(B)=v_{M_\lambda}(Q(A))$, then $v_{M_\lambda}(P(B))=v_{M_\lambda}(P(A))$.

Proof. The proof is the same as that of lemma 7 of the previous chapter, except that the induction involves all standard wffs P(A), satisfying the property that A does not occur in any predicate used to restrict variables, as well as satisfying the other conditions.

Lemma 9.

Let A and C be classes. If $v_{M_\lambda}(A \epsilon C)=1$ or 0 then $A \notin C$ has a gen. dep. set without any wffs of the form $A \notin B$ for any class B, except for A and for $B \in D^S$. The gen. dep. sets so constructed are such that (ii) is not applied to any atomic wffs of the form $A' \in A$.

Proof. The proof is the same as that of lemma 8 of the previous chapter.

Lemma 10.

Let A and B be classes. If $Y\epsilon A \leftrightarrow Y\epsilon B$ is valid in M_λ, then $A\epsilon A \leftrightarrow B\epsilon B$ has the value 1 in M_λ.

Proof. The proof is the same as that of lemma 9 of the previous chapter, except in (ii)(B) where the variable 'f' should be used instead of 'x'.

Theorem 4.

The Axiom of Extensionality (E) is valid in M_λ.

Proof. The proof is the same as that of Theorem 4 of the previous chapter, except in (B)(ii) where the variable 'f' should be used instead of 'x'.

Theorem 5.

$(Az)(z\epsilon f \leftrightarrow z\epsilon F) \supset (Ag)(f\epsilon g \leftrightarrow F\epsilon g)$ is valid in M_λ. (I.e. General Axiom 2 is valid in M_λ.)

Proof. (i) Let $A\notin D^S$. Let $z\epsilon a \leftrightarrow z\epsilon A$ be valid in M_λ, for some special class a. Hence $\sim z\epsilon a \vee z\epsilon A . \& . \sim z\epsilon A \vee z\epsilon a$ is valid in M_λ. Hence $v_{M_{\lambda+1}^S}(A\epsilon c) = v_{M_0}(a\epsilon c)$ and $v_{M_\lambda}(A\epsilon c \leftrightarrow a\epsilon c) = 1$, for any special class $c\epsilon D^S$.

(ii) Let $A\epsilon D^S$. Then, by the Axiom of Extensionality for special classes, the theorem holds.

Theorem 6.

(i) $Cl(X) \supset S(Y\epsilon X)$,

(ii) $Cl(X) \vee Cl(Y) \supset \sim S(XoY)$,

(iii) $FSCl(F) \supset F(F\epsilon f)$,

(iv) $PSCl(F) \supset P(F\epsilon f)$,

are all valid in M_λ. (I. e. General Axioms 5, 8, 3 and 4 are valid in M_λ.)

<u>Proof.</u> (i) and (ii) are valid by the definitions of the M_μ's. Let $v_{M_\lambda}(SCl(A)=0$, where A is a class. Then $v_{M_\lambda}((Af)(Sz)(z\epsilon f \;\&\; {\sim}z\epsilon A . v. z\epsilon A \;\&\; {\sim}z\epsilon f))=1$ and $v_{M_{\lambda+1}}(A\epsilon b)=0$, for any special class b. Hence (iii) is valid in M_λ.

Let $v_{M_\lambda}(A\epsilon b)=1$ or 0, for some special class b and some class A. Then either $A\notin D^S$, or $z\epsilon a \leftrightarrow z\epsilon A$ is valid in M_λ for some special class a, or $(Af)(Sz)(z\epsilon f \;\&\; {\sim}z\epsilon A . v. z\epsilon A \;\&\; {\sim}z\epsilon f))=1$ in M_λ. Hence $SCl(A)$ has the value 1 or 0 in M_λ. Hence if $v_{M_\lambda}(SCl(A))=\frac{1}{2}$ then $v_{M_\lambda}(A\epsilon b)=\frac{1}{2}$. Hence (iv) is valid in M_λ.

<u>Theorem 7.</u>

$(AX)\phi(X) \rightarrow (Ax)\phi(x)$ is valid in M_λ. (I.e. General Axiom 1 is valid in M_λ.)

<u>Proof.</u> Let $v_{M_\lambda}((AX)\phi(X))=1$. Then $v_{M_\lambda}(\phi(X))=1$, for all $X\epsilon D$. Hence $v_{M_\lambda}(\phi(x))=1$, for all $x\epsilon D^S$, since $D^S \subseteq D$. Therefore, $v_{M_\lambda}((Ax)\phi(x))=1$. Let $v_{M_\lambda}((AX)\phi(X))=\frac{1}{2}$. Then $v_{M_\lambda}(\phi(X))=\frac{1}{2}$ or 1, for all $X\epsilon D$. Hence $v_{M_\lambda}(\phi(x))=\frac{1}{2}$ or 1, for all $x\epsilon D^S$. Therefore, $v_{M_\lambda}((Ax)\phi(x))=\frac{1}{2}$ or 1. Let $v_{M_\lambda}((AX)\phi(X))=0$. Then $v_{M_\lambda}(\phi(X))=0$, $\frac{1}{2}$ or 1, for all $X\epsilon D$. Hence $v_{M_\lambda}(\phi(x))=0$, $\frac{1}{2}$ or 1, for all $x\epsilon D^S$. Hence $v_{M_\lambda}((Ax)\phi(x))=0$, $\frac{1}{2}$ or 1. Hence $(AX)\phi(X) \rightarrow (Ax)\phi(x)$ is valid in M_λ.

<u>Theorem 8.</u>

$k=1 \supset . k\epsilon F \leftrightarrow 1\epsilon F$ is valid in M_λ. (I.e. General Axiom 9 is valid in M_λ.)

<u>Proof.</u> The proof is similar to that of Theorem 4, only using individuals instead of classes. According to the definition of D(P), it can have members, AoB, where A and B are individuals, since AoB can take the value 1 or 0 in M_λ. In showing the validity of the Axiom of Extensionality, we were substituting one class for another and so there was no need to consider such atomic wffs as AoB. In this theorem, however, we need to take such atomic wffs into account. Lemmas 4 and 5 follow if A and B are individuals. Lemma 8 follows if A and B are individuals, where (ii) cannot be applied to any atomic wff of the form $C\epsilon A$ because $C\epsilon A$ is non-significant. Lemma 9 follows if A is an individual and C is a class, where $A\epsilon C$ has a gen. dep. set without any wffs of the form $A\epsilon B$ for any class B, except for $B\epsilon D^S$. The gen. dep. set is such that (ii) cannot be applied to any wffs of the form $A'\epsilon A$. Obviously there is no equivalent of lemma 10. The proof of Theorem 8 is similar to that of Theorem 4 except for the following :

(i) Instead of assuming that $V\epsilon A \leftrightarrow V\epsilon B$ is valid in M_λ, assume that A and B are individuals such that A=B has the value 1 in M_λ, i.e. $boA \leftrightarrow boB$ has the value 1 in M_λ, for all individuals b.

(ii) The only occurrences of A in D' are of the forms : $A\epsilon B'$, where $B'\epsilon D^S$ (B' being a special class), and C'oA and AoC", where C' and C" are individuals. $B\epsilon B' \leftrightarrow A\epsilon B'$ takes the value 1 in M_λ since Individual Axiom 4 is valid in M_λ. $C'oA \leftrightarrow C'oB$ and $AoC'' \leftrightarrow BoC''$ (also $AoA \leftrightarrow BoB$) take the value 1 in M_λ because of the above assumption. Hence, if $Q(A) \in D'$, $v_{M_\lambda}(Q(B)) = v_{M_\lambda}(Q(A))$. Hence, as in Theorem 4, $v_{M_\lambda}(B \notin C) = v_{M_\lambda}(A \notin C)$ and Theorem 8 follows.

Thus we have shown that M_λ is a model for the complete set of axioms given for the 4-valued theory of classes and individuals. This provides a proof of consistency, relative to Z-F, since D^S and D are sets.

The above method can be used to extend any set or class theory with a 3-valued model such that, (i) the third value n is characterised by the fact that $v(X \in \text{a class}) \neq n$ and $v(X \in \text{an individual}) = n$, for all X in the domain, that is, it divides the domain into two, (ii) the Axiom of Extensionality holds for classes, X and Y, i.e. if $v(W \in X) = v(W \in Y)$ for all W in the domain, then $v(X \in Z) = v(Y \in Z)$ for all classes Z of the domain, and (iii) a type of Axiom of Extensionality holds for individuals, X and Y, i.e. if X and Y are identical, . the sense that they are inter-substitutible in any context of the theory of individuals (if there is one), then $v(X \in Z) = v(Y \in Z)$, for all classes Z of the domain.

The extension becomes a 4-valued theory of classes and individuals satisfying the Axiom of Abstraction and the Axioms of Extensionality for both classes and individuals. The domain of this 4-valued theory is an extension of the domain of the 3-valued theory by adding extra classes, but no individuals. The 4-valued theory preserves the property of the value n.

If the independence of the Axiom of Choice, the Generalised Continuum Hypothesis and the Axiom of Constructibility can be shown by using inner models of the special class theory, as suggested in Chapter 4, then they would also be independent in the 4-valued theory.

The connectives and quantifiers of the 4-valued logic used to define standard wffs can be extended to include any which satisfy the following property, P :

(i) <u>For connectives, $\Gamma(p_1,\dots,p_n)$.</u>

Let X_0 be the set of indices i such that $v_M\ (p_i)=0$. Let X_1 be the set of indices i such that $v_M\ (p_i)=1$. Let X_n be the set of indices i such that $v_M\ (p_i)=n$. For some structure M', let X_0' be the set of indices i such that $v_{M'}(q_i)=0$, let X_1' be the set of indices i such that $v_{M'}(q_i)=1$, and let X_n' be the set of indices i such that $v_{M'}(q_i)=n$. Let $X_0 \subseteq X_0'$, $X_1 \subseteq X_1'$, and $X_n=X_n'$. Then, if $v_M(\Gamma(p_1,\dots,p_n))=k$ (1, 0 or n) then $v_{M'}(\Gamma(q_1,\dots,q_n))=k$, and if $v_{M'}(\Gamma(q_1,\dots,q_n))=n$ then $v_{M'}(\Gamma(p_1,\dots,p_n))=n$.

(ii) <u>For quantifiers, (QX)A(X).</u>

Let X_0 be the set of X's in D such that $v_M(A(X))=0$, let X_1 be the set of X's in D such that $v_M(A(X))=1$, and let X_n be the set of X's in D such that $v_M(A(X))=n$. For some structure M', let X_0' be the set of X's in D such that $v_{M'}(B(X))=0$, let X_1' be the set of X's in D such that $v_{M'}(B(X))=1$, and let X_n' be the set of X's in D such that $v_{M'}(B(X))=n$. Let $X_0 \subseteq X_0'$, $X_1 \subseteq X_1'$ and $X_n=X_n'$. If $v_M((QX)A(X))$

$=k$ (1, 0 or n) then $v_{M'}((QX)B(X))=k$, and if $v_{M'}((QX)B(X))=n$ then $v_M((QX)A(X))=n$.

(iii) For quantifiers, $(Qx)A(x)$.

This is similar to (ii) except D^S for D.

Proposition.

Any connective or quantifier defined in terms of connectives and quantifiers satisfying the property P also satisfies the property P.

Proof. (i) Connectives.

Let $v_M(\Gamma(\Delta_1(q_1,\ldots,q_n),\ldots,\Delta_m(q_1,\ldots,q_n)))=k$ (1, 0 or n), where Γ, $\Delta_1,\ldots,\Delta_m$ satisfy the property. Let X_0 be the set of indices i such that $v_M(q_i)=0$. Let X_1 be the set of indices i such that $v_M(q_i)=1$. Let X_n be the set of indices i such that $v_M(q_i)=n$. For some structure M', let X_0', X_1', X_n' be the corresponding sets of indices i for $r_1,\ldots,r_n$. Let $X_0\subseteq X_0'$, $X_1\subseteq X_1'$ and $X_n=X_n'$. Let Y_0, Y_1, Y_n be the sets of indices for $\Delta_i(q_1,\ldots,q_n)$ evaluated in M. Let $i\in Y_0\cup Y_1\cup Y_n$. Since $\Delta_i(q_1,\ldots,q_n)$ satisfies the property and $X_0\subseteq X_0'$, $X_1\subseteq X_1'$ and $X_n=X_n'$, $v_{M'}(\Delta_i(r_1,\ldots,r_n))=v_M(\Delta_i(q_1,\ldots,q_n))$. Hence if Y_0', Y_1', Y_n' are the sets of indices for $\Delta_i(r_1,\ldots,r_n)$ evaluated in M', then $Y_0\subseteq Y_0'$, $Y_1\subseteq Y_1'$ and $Y_n\subseteq Y_n'$. If $i\in Y_n'$, then $v_{M'}(\Delta_i(r_1,\ldots,r_n))=n$ and, since $X_0\subseteq X_0'$, $X_1\subseteq X_1'$ and $X_n=X_n'$, $v_M(\Delta_i(q_1,\ldots,q_n))=n$. Hence $Y_n'\subseteq Y_n$. Since $\Gamma(\Delta_1,\ldots,\Delta_m)$ satisfies the property, $v_{M'}(\Gamma(\Delta_1,\ldots,\Delta_m))$, with r's for q's, $=k$. Also if $v_{M'}(\Gamma(\Delta_1,\ldots,\Delta_m))=n$ then $v_M(\Gamma(\Delta_1,\ldots,\Delta_m))=n$.

(ii) Quantifiers.

Let $v_M(\Gamma((QX)\Delta(A(X))))=k$ (1, 0 or n), where Γ, Δ and (QX) satisfy the property P. Let X_0, X_1, X_n be the sets of X's in D for A(X) evaluated in M. Let X_0', X_1', X_n' be the sets of X's in D for B(X) evaluated in M'. Let $X_0 \subseteq X_0'$, $X_1 \subseteq X_1'$ and $X_n = X_n'$. Because Δ satisfies the property, if $v_M(\Delta(A(X)))=k$ (1, 0 or n) then $v_{M'}(\Delta(B(X)))=k$ and if $v_{M'}(\Delta(B(X)))=n$ then $v_M(\Delta(A(X)))=n$, for any $X \notin D$. If Y_0, Y_1 and Y_n are the sets of X's in D for $\Delta(A(X))$ evaluated in M and Y_0', Y_1' and Y_n' are the sets of X's in D for $\Delta(B(X))$ evaluated in M', then $Y_0 \subseteq Y_0'$, $Y_1 \subseteq Y_1'$ and $Y_n = Y_n'$. Because (QX) satisfies the property, if $v_M((QX)\Delta(A(X)))=k'$ (1, 0 or n) then $v_{M'}((QX)\Delta(B(X)))=k'$ and if $v_{M'}((QX)\Delta(B(X)))=n$ then $v_M((QX)\Delta(A(X)))=n$. Since Γ satisfies the property, $v_{M'}(\Gamma((QX)\Delta(B(X))))=k$ and if $v_{M'}(\Gamma((QX)\Delta(B(X))))=n$ then $v_M(\Gamma((QX)\Delta(A(X))))=n$.

Similarly for quantifiers (Qx).

Informally the conditions on connectives satisfying the property P are :

For the one-place connective $\Gamma(p)$, $\Gamma(n)=1$, $\frac{1}{2}$, 0 or n, if $\Gamma(1)=n$ or $\Gamma(\frac{1}{2})=n$ or $\Gamma(0)=n$ then $\Gamma(1)=n$ and $\Gamma(\frac{1}{2})=n$ and $\Gamma(0)=n$, if $\Gamma(\frac{1}{2})=1$ (or 0) then $\Gamma(1)=1$ (or 0) and $\Gamma(0)=1$ (or 0), and if $\Gamma(\frac{1}{2})=\frac{1}{2}$ then $\Gamma(1)=1$, 0 or $\frac{1}{2}$ and $\Gamma(0)=1$, 0 or $\frac{1}{2}$.

For the two-place connective $\Gamma(p,q)$, in each of the four boxes (bounded by continuous lines) in the diagram, there are either all

(p,q)	1	$\frac{1}{2}$	0	n
1				
$\frac{1}{2}$				
0				
n				

n's or no n's at all. If any value 1 (or 0) appears on the dotted line then there must be 1 (or 0) in the two places on either side of it and perpendicular to the dotted line, i.e. if $\Gamma(\frac{1}{2},0)=1$ then $\Gamma(1,0)=1$ and $\Gamma(0,0)=1$, and if $\Gamma(n,\frac{1}{2})=0$ then $\Gamma(n,1)=0$ and $\Gamma(n,0)=0$. If $\Gamma(\frac{1}{2},\frac{1}{2})=1$ then the whole box containing the $(\frac{1}{2},\frac{1}{2})$ position has 1's only in it. $\Gamma(n,n)$ can be any value.

The quantifier property is satisfied by (AX), (SX), (Ax), (Sx) and restricted quantifiers to A(X) or A(x), where the predicate A takes only the values 1, 0 and n ard takes them independently of the structures, M and M'.

To show that any of the above connectives and quantifiers can be used in defining standard wffs and hence be used in the Abstraction Axiom, it is only necessary to examine lemmas 1, 4 and 5 in the proof. Lemmas 1 and 4 are obvious from the definition of the property P. In lemma 5, replace the steps for connectives and quantifiers by the following :

(i) Let P(A) be $\Gamma(R_1(A),\dots,R_n(A))$. Let $v_M\ (\Gamma(R_1(A),\dots,R_n(A)))=1$ or 0 and let P(A) be W. Then $v_{M_{\nu_W}}(W)=1$ or 0. Let X_0, X_1 and X_n be the sets of indices i for $R_i(A)$ evaluated in M_{ν_W}. Let $i \notin X_0 \cup X_1$. Then $\nu_{R_i(A)} \leq \nu_W$, $D(R_i(A)) \subseteq D(W)$ and, using the lemma condition and ind. hyp., $v_{M_\lambda}(R_i(A)) = v_{M_\lambda}(R_i(B))$. Let X_0', X_1', X_n be the sets of indices i for $R_i(B)$ evaluated in M_λ. Hence $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. If $v_{M_{\nu_W}}(R_i(A))$ $=n$, then by lemma 4, $v_{M_\lambda}(R_i(B))=n$. If $v_{M_\lambda}(R_i(B))=n$, then $v_{M_{\nu_W}}(R_i(A))$ $=n$ and hence $X_n = X_n'$. By the property P for Γ, $v_{M_\lambda}(\Gamma(R_1(B),\dots,R_n(B)))$ $= v_{M_{\nu_W}}(\Gamma(R_1(A),\dots,R_n(A)))$ and $v_{M_\lambda}(P(B)) = v_{M_\lambda}(P(A))$.

(ii) Let $P(A)$ be $(QZ)R(A,Z)$ and $v_{M_\lambda}((QZ)R(A,Z))=1$ or 0. Let $P(A)$ be W. Let X_0, X_1 and X_n be the sets of Z's in D for $R(A,Z)$ evaluated in M_{ν_W}. Let $Z \notin X_0 \cup X_1$. Then $\nu_{R(A,Z)} \leq \nu_W$ and $D(R(A,Z)) \subseteq D(W)$. By the lemma condition and ind. hyp., $v_{M_\lambda}(R(B,Z)) = v_{M_\lambda}(R(A,Z))$. Let X_0', X_1' and X_n' be the sets of Z's for $R(B,Z)$ evaluated in M_λ. Hence $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$. If $v_{M_{\nu_W}}(R(A,Z))=n$ then, by lemma 4, $v_{M_\lambda}(R(B,Z))$ $=n$. If $v_{M_\lambda}(R(B,Z))=n$ then $v_{M_{\nu_W}}(R(A,Z))=n$. Hence $X_n = X_n'$. By the property of (QZ), $v_{M_\lambda}((QZ)R(B,Z)) = v_{M_{\nu_W}}((QZ)R(A,Z))$. Hence $v_{M_\lambda}(P(B))$ $= v_{M_\lambda}(P(A))$.

(iii) The case for $(Qz)R(A,z)$ is similar to (ii).

There is a further generalisation which allows any set or class theory using a many-valued (finite or infinite) logic L containing a value n and with a denumerable model N, in which the Axioms of

Extensionality for both classes and individuals are satisfied and $v(X \in \text{a class}) \neq n$ and $v(X \in \text{an individual}) = n$ for all X in the domain, to be extended to a class theory using a logic L' with one more value (call it 'pd') and with a model in which the Axioms of Abstraction and Extensionality for both classes and individuals are satisfied and in which the property of the value n is preserved.

Let the many-valued logic L contain designated values and undesignated values, where the value n is undesignated. The many-valued logic L must contain a quantifier S such that (Sz)A(z) takes the value k (one of the designated values) iff at least one of the A(z)'s are designated, takes the value n iff all of the A(z)'s take the value n, and takes the value m (one of the undesignated values (not n)) otherwise. The logic L must also contain a quantifier A such that (Az)A(z) takes the value k (the same value as above) iff all the A(z)'s are designated, takes the value n iff at least one of the values of the A(z)'s is n, and takes the value m (the same value as above) otherwise. The logic must contain an equivalence connective $\leftrightarrow$ such that $p \leftrightarrow q$ is designated iff p and q take the same value, takes the value n iff either p or q, but not both, takes the value n, and is undesignated (not n) otherwise. The logic contains an implication connective $\supset$ such that $p \supset q$ is designated iff q is designated or p is undesignated. It also contains a significance operator S such that Sp is designated iff p does not take the value n and is undesignated (not n) iff p takes the value n.

Also a method of restricting variables in the logic L is required. For this, see the method used in L'.

The many-valued logic L', which has an extra value (call it 'pd') added to L, must contain appropriate extensions of the quantifiers S and A and of the connectives $\leftrightarrow$, $\supset$ and S. The value pd is undesignated. If A(Z) takes a designated value for some Z, then (SZ)A(Z) takes the value k (as above) and otherwise if A(Z) takes the value pd for some A(Z) then (SZ)A(Z) takes the value pd. If A(Z) takes the value n for some Z then (AZ)A(Z) takes the value n, if A(Z) takes no values n for any Z and A(Z) is undesignated (not n or pd) for some Z, then (AZ)A(Z) takes the value m (as above), and otherwise if A(Z) takes the value pd for some Z, then (AZ)A(Z) takes the value pd. $p \supset q$ is designated iff q is designated or p is undesignated. $p \leftrightarrow q$ takes the value pd if either p or q takes the value pd except when both p and q take the value pd in which case $p \leftrightarrow q$ is designated, and except when the other one takes the value n in which case $p \leftrightarrow q$ takes the value n. Sp is designated if p takes the value pd. $\supset$ and &' can be used to restrict variables, where p &' q is designated if both p and q are designated, undesignated if p is designated and q is undesignated, and takes the value n if p is undesignated.

The Abstraction Axiom can be stated as $(AX,z_1,\ldots,z_m,Z_1,\ldots,Z_n)$ $S\emptyset(X,z_1,\ldots,z_m,Z_1,\ldots,Z_n) \supset (SY)(AX)(X\in Y \leftrightarrow \emptyset(X,z_1,\ldots,z_m,Z_1,\ldots,Z_n))$, where $\emptyset$ is constructed using the connectives and quantifiers used in forming standard wffs. The Extensionality Axiom for classes, F and G, can be stated as $(AZ)(Z\epsilon F \leftrightarrow Z\epsilon G) \supset (AH)(F\epsilon H \leftrightarrow G\epsilon H)$. The Extensionality Axiom for special classes, f and g, can be stated as $(Az)(z\epsilon f \leftrightarrow z\epsilon g) \supset (Ah)(f\epsilon h \leftrightarrow g\epsilon h)$. The Extensionality Axiom for individuals, k and l, can be stated as $k=l \supset (AF)(k\epsilon F \leftrightarrow l\epsilon F)$, where $k=l$ is an identity defined in the theory of individuals and special classes such that k and l, being identical, can be inter-substituted in all contexts. In particular, the Extensionality Axiom for individuals, k and l, must hold in the theory of individuals and special classes, i.e. $k=l \supset (Af)(k\epsilon f \leftrightarrow l\epsilon f)$.

$SCl(F)$ is defined as $(Sf)(Az)(z\epsilon f \leftrightarrow z\epsilon F)$.

Note that all the symbolism of the 3- and 4-valued theories are carried over to the many-valued theories.

The propositional constants are omitted from the atomic wffs and if atomic wffs with some of these values are wanted then perhaps an atomic wff of the form a b can be used. The connectives and quantifiers used in forming standard wffs are ones which satisfy the property S :

(i) For connectives $\Gamma(p_1,\ldots,p_n)$.

Let X_m be the set of indices i such that $v_M(p_i)=m$, for each value m of L. For some structure M', let X_m' be the set of indices i such that $v_{M'}(q_i)=m$, for each value m of L. If $X_m \subseteq X_m'$ for all values m of L and $X_n=X_n'$ for the value n of L, then if $v_M(\Gamma(p_1,\ldots,p_n))=k$ (some value of L) then $v_{M'}(\Gamma(q_1,\ldots,q_n))=k$ and if $v_{M'}(\Gamma(q_1,\ldots,q_n))$ $=n$ then $v_M(\Gamma(p_1,\ldots,p_n))=n$.

(ii) For quantifiers $(QX)A(X)$.

Let X_m be the set of X's in D such that $v_M(A(X))=m$, for each value m of L. For some structure M', let X_m' be the set of X's in D such that $v_{M'}(B(X))=m$, for each m in L. If $X_m \subseteq X_m'$ for all m of L and $X_n=X_n'$ for the value n of L, then if $v_M((QX)A(X))=k$ (some value of L) then $v_{M'}((QX)B(X))=k$ and if $v_{M'}((QX)B(X))=n$ then $v_M((QX)A(X))=n$.

(iii) For quantifiers $(Qx)A(x)$.

This is similar to (ii), except D^S for D.

Similarly to the 4-valued case, it can be shown that any connective or quantifier defined in terms of connectives and quantifiers satisfying the property S also satisfies the property S. The quantifier property S is satisfied by the quantifiers S and A, unrestricted or restricted to a predicate A(X) or A(x), where A does not take the value pd and takes its values independently of the structures, M and M'.

We now show that the Axioms of Abstraction and Extensionality (for both classes and individuals) hold in the extended system with logic L'.

$M_1 \leq M_2$ is defined as : $M_1 \leq M_2$ iff, for any atomic wff P, if $v_{M_1}(P)=m$, for some value m of L, then $v_{M_2}(P)=m$, and if $v_{M_2}(P)=n$ then $v_{M_1}(P)=n$.

Lemma 1 follows by the definition of standard wff. M_0 is defined as follows :

If $A \notin D^S$ or $B \notin D^S$, then $v_{M_0}(A \in B)=pd$, where B is a class, or $v_{M_0}(A \in B)=n$, where B is an individual. If $A \in D^S$ and $B \in D^S$ then $v_{M_0}(A \in B)=$ the value of L given to $A \epsilon B$ in the model N of the theory of individuals and special classes.

Assuming M_μ defined for some ordinal μ, $M_{\mu+1}$ is defined as follows : $v_{M_{\mu+1}}(A \epsilon \{X : P(X)\}) = v_{M_\mu}(P(A))$. If b is an individual then $v_{M_{\mu+1}}(A \epsilon b) = n$. Now let b be a special class and $A \epsilon D - D^S$. If $v_{M_\mu}(z \epsilon A) = v_{M_\mu}(z \epsilon a)$ for all $z \epsilon D^S$, for some $a \epsilon D^S$, then $v_{M_{\mu+1}}(A \epsilon b) = v_{M_0}(a \epsilon b)$. If there is no $a \epsilon D^S$ such that, for all $z \epsilon D^S$, $v_{M_\mu}(z \epsilon A) = v_{M_\mu}(z \epsilon a)$, then $v_{M_{\mu+1}}(A \epsilon b) = v_{M_\mu}(SCl(A))$. Note that SCl(F) has the property S, because $z \epsilon f$ only takes values in L. Also $v_{M_\mu}(z \epsilon A) = v_{M_\mu}(z \epsilon a)$ for all $z \epsilon D^S$, for some $a \epsilon D^S$ iff $v_{M_\mu}(SCl(A))$ is designated, and there is no $a \epsilon D^S$ such that, for all $z \epsilon D^S$, $v_{M_\mu}(z \epsilon A) = v_{M_\mu}(z \epsilon a)$ iff $v_{M_\mu}(SCl(A))$ is undesignated. If μ is a limit ordinal, on the assumption that $M_\nu \leq M_\tau$ for all $\nu \leq \tau$, for all $\tau < \mu$, for all atomic wffs P, if $v_{M_\nu}(P) = k$, for some value k in L, for some $\nu < \mu$, then $v_{M_\mu}(P) = k$, and if $v_{M_\nu}(P) = pd$ for all $\nu < \mu$, then $v_{M_\mu}(P) = pd$.

If there is any relation, R(k,l) say, among individuals defined in the theory of individuals and special classes, then $v_{M_\mu}(R(k,l)) =$ the value of L given to R(k,l) in the model N, and $v_{M_\mu}(R(A,k)) = v_{M_\mu}(R(k,A)) = v_{M_\mu}(R(A,B)) = n$, where A and B are classes. The values are the same for all structures M_μ.

Lemma 2 follows similarly to before. In case (c), let $v_{M_\nu}(A \epsilon b) = k$ for some value k ($\neq n$) of L. Then there is an ordinal $\eta < \nu$ such that $v_{M_\eta}(SCl(A)) = m$, for some value m in L. Since $\eta \leq \mu - 1$, $M_\eta \leq M_{\mu-1}$. Hence $v_{M_{\mu-1}}(SCl(A)) = m$. If m is undesignated, m=k and $v_{M_\mu}(A \epsilon b) = k$. If m is designated, then there is an $a \in D^S$ such that $v_{M_0}(z \in a) = v_{M_\eta}(z \in A)$ for all $z \in D^S$. Then $v_{M_0}(a \in b) = k$. Hence $v_{M_{\mu-1}}(z \in A) = v_{M_0}(z \in a)$, for all $z \in D^S$, and $v_{M_\mu}(A \in b) = k$.

Lemma 3 follows similarly to before except that there is one increasing chain of subsets of the denumerable set of all atomic wffs for every value in L except n. Theorem 1 follows similarly to before, but shows that any wff valid in the model N of the theory of individuals and special classes is valid in M . Theorems 2 and 3 follow as before. The definitions of ν_P and D(P) are the same except that all values of L except n must be put in place of the values 1 and 0. Lemmas 4 and 5 can be shown by replacing X_0 and X_1 by X_k, for all values k of L except n. Corresponding standard wff and general dependent set are defined similarly. Lemmas 6, 7, 8 and 9 follow by replacing 1 and 0 by the values of L except for n. In lemma 10, (ii)(A) becomes : Let SCl(A) be valid in M_λ. Then $v_{M_\lambda}(z \in A) = v_{M_\lambda}(z \in a)$, for all $z \in D^S$, for some $a \in D^S$. Hence $v_{M_{\lambda+1}}(A \in B') = v_{M_0}(a \in B')$. By the condition of the lemma, $v_{M_\lambda}(z \in B) = v_{M_\lambda}(z \in a)$, for all $z \in D^S$. Hence $v_{M_{\lambda+1}}(B \in B') = v_{M_0}(a \in B')$. Hence $v_{M_\lambda}(A \in B') = v_{M_\lambda}(B \in B')$.

(ii)(B) becomes : Let SCl(A) be invalid in M_λ. $v_{M_\lambda}(SCl(A)) \neq pd$ because $v_{M_\lambda}(A \in B')$ is a value of L. Hence $v_{M_{\lambda+1}}(A \in B') = v_{M_\lambda}(SCl(A))$. By the lemma condition, $v_{M_\lambda}(SCl(A)) = v_{M_\lambda}(SCl(B))$. Hence $v_{M_{\lambda+1}}(B \in B') = v_{M_\lambda}(SCl(A))$ and $v_{M_\lambda}(B \in B') = v_{M_\lambda}(A \in B')$. The rest of lemma 10 follows as before.

In Theorem 4, B(i) and B(ii) are similar to (ii)(A) and (ii)(B), respectively, of lemma 10. Otherwise the theorem follows as before.

Theorem 5 follows as before.

Theorem 6 needs two monadic operators : C such that Cp is designated iff p takes a value in L excepting n, and U such that Up is designated iff p is undesignated. Theorem 6 becomes : (i) $S(Y \in F)$ and (ii) $USCl(F) \supset . SCl(F) \leftrightarrow F \in f$, are valid in M_{λ}. The proof is obvious.

Theorem 7 becomes : $(AX)\emptyset(X) \supset (Ax)\emptyset(x)$ is valid in M_{λ}, which is obvious.

Theorem 8 follows similarly to before, except that account must be taken of individuals. If a relation, $R(k,l)$, is defined for individuals, then $k=l$ must imply that k and l are inter-substitutible in contexts involving this relation.

Hence appropriate generalisations of the usual theorems that hold in the 4-valued class theory hold in this many-valued theory.

CHAPTER 8.

SIGNIFICANCE RANGE THEORY.

(i) The Connectives and Quantifiers Used to Define Significance Ranges.

Firstly I want to consider the significance ranges that might arise from the 3-valued theory of classes and individuals of Chapter 4. By Axiom B, $(Ax_1',\ldots,x_\ell', y_1,\ldots,y_m)SS\phi(x_1',\ldots,x_\ell', y_1,\ldots,y_m) \supset (Sf\ )(Ax_1',\ldots,x_\ell')(\langle x_1',\ldots,x_\ell'\rangle\epsilon f \equiv S\phi(x_1',\ldots,x_\ell', y_1,\ldots,y_m))$, where quantification is over sets and individuals only. Since SSp is valid in the logic, $(Sf)(Ax_1',\ldots,x_\ell')(\langle x_1',\ldots,x_\ell'\rangle\epsilon f \equiv S\phi(x_1',\ldots,x_\ell', y_1,\ldots,y_m))$. Although this would seem to yield the most natural definition of a significance range from the Axiom B, this does not specify the class f uniquely. Hence I will use the following definition : The unique class f, such that $(Az')(z'\epsilon f \equiv (Sx_1',\ldots,x_\ell')(T(z'=\langle x_1',\ldots,x_\ell'\rangle) \ \& \ S\phi(x_1',\ldots,x_\ell', \bar{y}_1,\ldots,\bar{y}_m)))$, is a <u>significance range.</u> (ϕ, of course, contains quantification over sets and individuals only.) We shall also call this unique class f, <u>the significance range of $\phi(x_1',\ldots, x_\ell', \bar{y}_1,\ldots,\bar{y}_m)$.</u> Since there are no restrictions on the connectives that can be used to construct ϕ, the unique class f, such that $(Az')(z'\epsilon f \equiv (Sx_1',\ldots,x_\ell')(T(z'=\langle x_1',\ldots,x_\ell'\rangle) \ \& \ ST_n\phi(x_1',\ldots,x_\ell', \bar{y}_1,\ldots,\bar{y}_m)))$, is the significance range of $T_n\phi$. But $ST_np \equiv Tp$, and hence every class, uniquely defined as a class of ℓ-tuples for some predicate $\phi(x_1',\ldots,x_\ell', \bar{y}_1,\ldots,\bar{y}_m)$, which is significant for all substitutions into its free variables, is the significance range of $T_n\phi(x_1', \ldots, x_\ell', \bar{y}_1, \ldots, \bar{y}_m)$.*

This is an undesirable result as there are many examples in ordinary discourse of classes which are not significance ranges of any predicate, if one restricts the connectives used to construct predicates so that the predicates so formed can be interpreted in ordinary discourse. A simple example of such a class is the class consisting of a single member, say, a particular leaf of a tree or a number. Unless T_n is used in restricting the S-quantifier, it cannot be used to construct a predicate which can be interpreted in ordinary discourse. T_n is an operator which can convert a false proposition to a non-significant one, so there must be something intrinsically non-significant about T_n. T_n has no interpretation on its own and it was only introduced to serve the purpose of restricting the S-quantifier and such restricting does not give an interpretation to T_n.

The difference between classes and significance ranges on this point is that, by Theorem 1 of Chapter 4, all classes can be generated by predicates constructed using only the connectives $\sim$, & and T and the quantifier A, whereas by introducing further connectives, such as T_n, one may define significance ranges which would not have been definable had these further connectives not been added. It remains to determine what connectives and quantifiers should be used in constructing predicates to generate significance

(*): See also [23a],

ranges. One of the requirements is that the predicates constructed using these connectives and quantifiers must be such that they can be interpreted in ordinary discourse. This should also apply to predicates used to generate classes, but the only reasons for allowing all connectives and quantifiers is that it simplifies the formal treatment not to place restrictions on them and no classes are formed which could not have been formed by using predicates with some interpretation in ordinary discourse. If there is no such interpretation of a predicate then there would be no such interpretation of the class generated by it nor of its significance range.

The connectives, ~, & and T, can be interpreted as 'not', 'and' and 'it is true that', respectively. As explained in Chapter 1, v can be used to formally construct 'fx v gx', which has the same value as '(f or g)x', which can be interpreted in ordinary discourse as a predicate disjunction. The example given was 'x is a holiday or likes cheese'. The quantifier A can be interpreted as 'for all'. As explained in Chapter 1, the quantifier S can be interpreted as 'for some' as in the example, 'Something is happy' or 'For some x, x is happy'. As also pointed out, v can also be used to express S-quantification over a finite range. Given a predicate A(x) such that A(x) is true for some x, the quantifier S restricted by the predicate A can be interpreted as 'for some x such that A(x)'. Formally this is represented as $(Sx)(T_nA(x)$

& $\phi(x))$, where ϕ is the predicate which is quantified. Given a similar predicate $A(x)$, the quantifier A restricted by the predicate A can be interpreted as 'for all x such that $A(x)$'. Formally this is represented as $(Ax)(A(x) \supset \phi(x))$, where ϕ is the predicate which is quantified. The connective S can be interpreted as 'it is significant that'. The connective $\supset$, as well as being used to restrict the A-quantifier, can be interpreted as 'if ~~... is true,~~ it is true that ..., then ...'. This has to be interpreted in a similar way to the material implication of the 2-valued propositional calculus, in that if the antecedent is not true then the implicational statement is vacuously true and if the antecedent is true then the implicational statement takes the value of the conclusion.

However, peculiar significance ranges can be formed by using the connective $\supset$. Form the significance range f such that $(Az')(z' \in f \equiv S(\phi_1(z') \supset \phi_2(z')))$. Hence $(Az')(z' \in f \equiv {\sim}T\phi_1(z') \text{ v. } T\phi_1(z')$ & $S\phi_2(z'))$. Thus, f is the union of the ~T-range of ϕ_1 and the intersection of the T-range of ϕ_1 and the significance range of ϕ_2. f is a peculiar sort of construct from these ranges. This is a similar situation to that in the Introduction where the Lukasiewicz $\rightarrow$ is rejected as a connective used in constructing predicates to generate classes. In the above example if the significance range of ϕ_2 is empty (which can sometimes occur, e.g. as the intersection of two disjoint significance ranges) then the significance range f is the ~T-range of ϕ_1, i.e. the union of the ~S-range of ϕ_1 and the F-range of ϕ_1. Take the example of the predicate x=2 in formal arithmetic. Its ~T-range consists of everything except the number 2. This can hardly be regarded as a significance range as it does not exhaust one sort of thing nor even many sorts of things since the number 2 must be the same sort of thing as the number 3, say.

So not all connectives used to construct predicates, which can be interpreted in ordinary discourse, can be used in the formation of significance ranges. The foregoing argument in the case of $\supset$ suggests that significance ranges should be constructed from other significance ranges rather than from T-ranges, F-ranges, ~T-ranges, and ~F-ranges. This does seem plausible enough since classes such as the one above consisting of everything except the number 2 should not be able to influence the construction of significance ranges. In order to satisfy this property the connectives and quantifiers must produce predicates whose significance depends only on the significance of the atomic formulae in the predicates. For example, the connective & satisfies the property because $S(p \,\&\, q) \equiv Sp \,\&\, Sq$. In fact, what is required is for the connectives and quantifiers to be able to be used to form an "s-n sublogic".

An s-n sublogic is obtained by grouping together the significant values, 1 and 0, and calling it the value s, while the non-significant value n remains intact. In order to be able to perform this on a connective or quantifier one must be able to consistently assign the value s or n in the 2-valued matrix of the connective and in the 2-valued description of the quantifier. One can do this for the connectives ~, &, v, T, S and the quantifiers A and S as follows :

~	
s	s
n	n

&	s	n
s	s	n
n	n	n

v	s	n
s	s	s
n	s	n

T	
s	s
n	s

S	
s	s
n	s

$(Ax)\phi(x)$ takes the value s if $\phi(x)$ takes the value s for all x and $(Ax)\phi(x)$ takes the value n if $\phi(x)$ takes the value n for some x. $(Sx)\phi(x)$ takes the value s if $\phi(x)$ takes the value s for some x and $(Sx)\phi(x)$ takes the value n if $\phi(x)$ takes the value n for all x.

However, there is no consistent assignment for $\supset$.

$\supset$	s	n
s	s	s or n
n	s	s

If p is significant and q is non-significant then $p \supset q$ is significant or non-significant according to whether p is false or p is true, respectively. This is in fact what caused the problem about significance ranges generated by predicates containing $\supset$.

However, consider the connective N defined by the matrix :

N	
1	n
0	n
n	1

N can be consistently assigned values in an s-n sublogic as follows :

N	
s	n
n	s

N satisfies the equivalence, $SNp \equiv \sim Sp$. Form the significance range f such that $(Az')(z' \in f \equiv SN\phi(z'))$. By the equivalence, $(Az')(z' \in f \equiv \sim S\phi(z'))$. So the significance range of $N\phi(z')$ is the ~S-range of $\phi(z')$. But not every ~S-range is a significance range. Consider the predicate 'x is prime'. This has a significance range consisting of all natural numbers. Its ~S-range would consist of all things which are not natural numbers. This can hardly be said to be a significance range because one cannot isolate the natural numbers from the other rational numbers or from the other real numbers so as to form a significance range without the natural numbers. Such a significance range would not exhaust one sort of thing or many sorts of things, without containing only a part of such a sort of thing. Hence the connective N, although it can consistently be assigned values in an s-n sublogic, cannot be used to form predicates which generate significance ranges.

The preceding argument suggests that the connectives and quantifiers which can be used to form an s-n sublogic should be positive so that the significance of any predicate constructed using them depends positively on the significance ranges of the atomic wffs, i.e. it does not depend on the ~S-ranges of any of the atomic wffs.

The s-n sublogic of such monadic and dyadic connectives is represented by the following matrices :

Monadic.

(1)
$$\begin{array}{c|c} & \\ \hline s & s \\ n & s \end{array}$$

(2)
$$\begin{array}{c|c} & \\ \hline s & s \\ n & n \end{array}$$

(3)
$$\begin{array}{c|c} & \\ \hline s & n \\ n & n \end{array}$$

Dyadic.

(1)
$$\begin{array}{c|cc} & s & n \\ \hline s & s & s \\ n & s & s \end{array}$$

(2)
$$\begin{array}{c|cc} & s & n \\ \hline s & s & s \\ n & s & n \end{array}$$

(3)
$$\begin{array}{c|cc} & s & n \\ \hline s & s & s \\ n & n & n \end{array}$$

(4)
$$\begin{array}{c|cc} & s & n \\ \hline s & s & n \\ n & s & n \end{array}$$

(5)
$$\begin{array}{c|cc} & s & n \\ \hline s & s & n \\ n & n & n \end{array}$$

(6)
$$\begin{array}{c|cc} & s & n \\ \hline s & n & n \\ n & n & n \end{array}$$

We will now show that ~, &, v and T and connectives defined in terms of them, assuming that the constants 1, 0 and n are obtainable from the theory of classes and individuals, will not only exhaust all of the above possibilities for monadic and dyadic connectives ~~and~~ but also exhaust all of the "positive" connectives which can be used to form an s-n sublogic.

Note that ~, &, v and T are all "positive" connectives which can be used to form an s-n sublogic. The classical connectives ~ and & will yield all the monadic connectives of type (2) and all the dyadic connectives of type (5). The constant n will yield the monadic connective of type (3) in the form SSp & n and the dyadic connective of type (6) in the form SSp & SSq & n. [Note that Sp is definable as Tp v T~p. Also Fp is definable as T~p.]

The monadic connectives of type (1) are obtained in the form (~Tp v (1 or 0)) & (~Fp v (1 or 0)) & (Sp v (1 or 0)), where substitutions of 1 or 0 in the places indicated will yield the eight required connectives. The dyadic connectives of type (1) are obtained in the form (~(Tp & Tq) v (1 or 0)) & (~(Tp & Fq) v (1 or 0)) & (~(Tp & ~Sq) v (1 or 0)) & (~(Fp & Tq) v (1 or 0)) & (~(Fp & Fq) v (1 or 0)) & (~(Fp & ~Sq) v (1 or 0)) & (~(~Sp & Tq) v (1 or 0)) & (~(~Sp & Fq) v (1 or 0)) & (~(~Sp & ~Sq) v (1 or 0)), where substitutions of 1 or 0 in the places indicated will yield the 2^9 required connectives.

Next consider the connective defined as (p v ~p) v (q v ~q). It has the matrix :

	1	0	n
1	1	1	1
0	1	1	1
n	1	1	n

Take each of the dyadic connectives of type (1) in turn and form the conjunction, using &, of it and (p v ~p) v (q v ~q). All the places in the matrix of the dyadic connective of type (1) will remain intact except for the n-n place, which will be converted to an 'n'. By using this method all the dyadic connectives of type (2) can be defined.

Next consider the connective defined as $(p \vee \sim p)$ & SSq. It has the matrix :

	1	0	n
1	1	1	1
0	1	1	1
n	n	n	n

As above, take each of the dyadic connectives of the type (1) in turn and form the conjunction, using &, of it and $(p \vee \sim p)$ & SSq. All the places in the matrix of the dyadic connective of type (1) will remain intact except for the n-1, n-0 and n-n places, which will be converted to 'n' 's. By using this method all the dyadic connectives of type (3) can be defined.

Similarly, by using the connective defined as SSp & $(q \vee \sim q)$, all the dyadic connectives of type (4) can be defined.

Hence ~, &, v and T are sufficient to define all the "positive" monadic and dyadic connectives which can be used to form an s-n sublogic. The next task is to generalise the above procedure to deal with n-adic connectives.

The type (1) monadic and dyadic connectives can be generalised to n-adic connectives by representing each place $(a_1, \ldots, a_n)$ in the matrix by a conjunction K_1p_1 & ... & K_np_n, where K_i is T, F or ~S according as a_i is 1, 0 or n, respectively, and by forming the conjunction of all expressions of the form (~(K_1p_1 & .... & K_np_n) v (1 or 0)), where substitutions of 1 and 0 in the places indicated will yield all the required connectives. By forming arbitrary disjunctions and conjunctions of the formulae $p_i \vee \sim p_i$, for $i=1,\ldots,n$, so that if $p_j \vee \sim p_j$ does not occur in it then the formula is conjoined with SSp_j, all the formulae generalising the formulae, $(p \vee \sim p) \vee (q \vee \sim q)$, $(p \vee \sim p)$ & SSq, SSp & $(q \vee \sim q)$, and $(p \vee \sim p)$ & $(q \vee \sim q)$ of the dyadic case, can be obtained. [We could have used $(p \vee \sim p)$ & $(q \vee \sim q)$ in an alternative method of obtaining the dyadic connectives of type (5).] In the case of $n=3$, $(p_1 \vee \sim p_1) \vee (p_2 \vee \sim p_2) \vee (p_3 \vee \sim p_3)$ has a single value n when p_1, p_2 and p_3 all take the value n, $((p_1 \vee \sim p_1) \vee (p_2 \vee \sim p_2))$ & SSp_3 has a "line" of values n when p_1 and p_2 take the value n, and $(p_1 \vee \sim p_1)$ & SSp_2 & SSp_3 has a "plane" of values n when p_1 takes the value n. By forming conjunctions of these formulae, one can form 2 or 3 "planes" of values n, 2 or 3 "lines " of values n, and a "plane" and "line" intersecting at (n,n,n). So, in general the disjunctions of the atomic elements $p_i \vee \sim p_i$ determine the simplex and the conjunctions of these disjunctions superimpose the simplexes to form the configuration of the n's.

All of these formulae take the value 1 whenever it is significant and so, using the same method as was used in the dyadic case, any significant value can replace a value 1 in any of these formulae. A formula taking the value n, whatever the values of its propositional variables, $p_1,\ldots,p_n$, can be defined as n & SSp_1 & ... & SSp_n. This completes the proof.

Hence ~, &, v, T and the constants 1, 0 and n generate all the "positive" connectives which can be used to form an s-n sublogic.

If a quantifier, when used over a finite domain, can be replaced by a "positive" connective which can be used in forming an s-n sublogic, then this quantifier can be defined in terms of A and S. Both A and S are "positive" because the significance of the quantified statements, $(Ax)\phi(x)$ and $(Sx)\phi(x)$, can be determined from the significance of the $\phi(x)$'s. The same applies for restricted quantification since $(Ax)(A(x) \supset \phi(x))$ takes the value s if $\phi(x)$ takes the value s for all x such that $A(x)$, $(Ax)(A(x) \supset \phi(x))$ takes the value n if $\phi(x)$ takes the value n for some x such that $A(x)$, $(Sx)(T_nA(x) \& \phi(x))$ takes the value s if $\phi(x)$ takes the value s for some x such that $A(x)$, and $(Sx)(T_nA(x) \& \phi(x))$ takes the value n if $\phi(x)$ takes the value n for all x such that $A(x)$.

Now that the connectives ~, &, v and T and the quantifiers A and S (restricted and unrestricted) have been characterised, let us see what significance ranges result from predicates constructed using them. Since $S{\sim}p \equiv Sp$, the significance range of $\sim\phi(z')$ is the same as that of $\phi(z')$. So no new significance ranges can be introduced by using ~.

Since STp is valid, the significance range of $T\phi(z')$ is the universal class of all sets and individuals. Intuitively, there should be a significance range of all classes and individuals determined by using a predicate such as 'x is a member of the null class (or any particular class)'. One of the drawbacks of the theory of Chapter 4 is that one can only form classes of sets and individuals and not classes of classes and individuals, in general. Accepting this drawback, the significance range of all sets and individuals, obtained above, is the closest one can get to the desired significance range of all classes and individuals.

The predicate $\phi(z')$ can also be a constant, taking the value 1, 0 or n. If it takes the value 1 or 0 then its significance range is the same as for $T\phi(z')$, as above. If it takes the value n then its significance range is empty. This is the same as would be produced by the intersection of two disjoint significance ranges, which will be considered next.

Since $S(p \,\&\, q) \equiv Sp \,\&\, Sq$, the significance range of $\phi_1(z')$ & $\phi_2(z')$ is the intersection of the significance ranges of $\phi_1(z')$ and $\phi_2(z')$. If these two significance ranges exhaust some sorts of things then their intersection exhausts the common sorts of things, if it consists of anything at all. For example, the predicate 'x' is blue and hard (physically firm)' has a significance range consisting of all material objects, which is the intersection of the significance ranges of all extended things and all material objects, these two significance ranges being determined by the predicates, ' x' is blue ' and ' x' is hard (physically firm) ', respectively. Consider also the predicate ' x' is blue and is a holiday'. Its significance range is empty because the significance ranges of ' x' is blue ' and ' x' is a holiday ' are disjoint. However, this empty significance range is quite all right because it is determined by the predicate ' x' is blue and is a holiday ' and because in establishing the notion of significance range it is awkward to try to avoid including it.

Since $S(p \vee q) \equiv Sp \vee Sq$, the significance range of $\phi_1(z') \vee \phi_2(z')$ is the union of the significance ranges of $\phi_1(z')$ and $\phi_2(z')$. If these two significance ranges exhaust some sorts of things then their union exhausts these sorts of things. For example, the predicate ' x' is blue or is a holiday ' has a significance range consisting of all extended things and all days, which is the union of the significance range of all extended things, determined by the predicate ' x' is blue ', and the significance range of all days, determined by the predicate ' x' is a holiday '.

Since $S(Ax')\phi(x',z') \equiv (Ax')S\phi(x',z')$, the significance range of $(Ax')\phi(x',z')$ is the intersection of the significance ranges of $\phi(x',z')$, for all x'. The significance range of $(Ax')\phi(x',z')$ would then exhaust all the common sorts of things present in all of the significance ranges of the $\phi(x',z')$'s, assuming these are ranges of 1-tuples. For example, the predicate ' x' is similar to everything ' or '(Ay')(x' is similar to y')' has an empty significance range because it is the intersection of many disjoint significance ranges of the predicates ' x' is similar to y' ', for all y'. However, the predicate ' x' likes everything ' has a significance range consisting of all animals because it is the intersection of identical significance ranges of the predicates ' x' likes y' ', for all y'.

Since $S(Sx')\emptyset(x',z') \equiv (Sx')S\emptyset(x',z')$, the significance range of $(Sx')\emptyset(x',z')$ is the union of the significance ranges of $\emptyset(x',z')$, for all x'. The significance range of $(Sx')\emptyset(x',z')$ would then exhaust all the sorts of things present in at least one of the significance ranges of the $\emptyset(x',z')$'s, assuming these are ranges of 1-tuples. For example, the predicate, ' x' is similar to something ' has a universal significance range because it is the union of the significance ranges of the predicates, ' x' is similar to y' ', for all y', and there is always something which is similar or not similar to x'. The predicate ' x' likes something ' has a significance range consisting of all animals because it is the union of identical significance ranges of the predicates, ' x' likes y' ', for all y'.

The significance ranges resulting from the use of restricted quantification are restricted unions and intersections of significance ranges, determined in a similar way to those obtained from using unrestricted quantification.

So, the connectives ~, &, v and T and the quantifiers, A and S (restricted and unrestricted) are satisfactory for the purpose of determining significance ranges. The predicates constructed using them can be interpreted in ordinary discourse. Because they are "positive" connectives which can be used to form an s-n sub-logic, the significance ranges of predicates formed using them depend only on the significance ranges of atomic predicates (and on the universal significance range). Hence, instead of significance range theory being just a theory of s-ranges of classes, according to the definition at the beginning of the chapter, significance range theory is an independent subtheory of the theory of classes. That is, significance ranges do not depend on T-ranges, F-ranges, ~T-ranges, ~F-ranges or even ~S-ranges. Significance ranges have their own characterisation as being classes whose members exhaust one sort of thing or exhaust some sorts of things, assuming that these classes are classes of 1-tuples. Significance ranges of n-tuples derive their character from the significance ranges of 1-tuples. Here, the empty and universal significance ranges must be included as well. Classes, on the other hand, can be made up of arbitrary members, where there is no necessity to exhaust a sort of thing just because one of that sort is a member.

The formal definition of significance range is as follows : The unique class f, such that $(Az')(z' \in f \equiv (Sx_1',\ldots,x_\ell')(T(z' = \langle x_1',\ldots,x_\ell' \rangle) \& S\emptyset(x_1',\ldots,x_\ell', \bar{y}_1,\ldots,\bar{y}_m)))$, where $\emptyset$ is constructed using only the connectives ~, &, v and T and only the quantifiers, A and S (restricted and unrestricted), is <u>the significance range of $\emptyset(x_1',\ldots,x_\ell', \bar{y}_1,\ldots,\bar{y}_m)$.</u>

Theorem.

If f is the significance range of ϕ, constructed as above, then there is a predicate ϕ', constructed using only the connectives & and v and only the quantifiers A and S(restricted and unrestricted), such that f is the significance range of ϕ'.

Proof. This is shown by induction on the number of connectives and quantifiers used in the construction of ϕ.

(1) It is clear in the case of atomic wffs.

(2) Let ϕ be $\sim\psi$. By the induction hypothesis, there is a ψ', constructed using only &, v, A, S, such that $S\psi \equiv S\psi'$. Since $S\phi \equiv S\psi$, $S\phi \equiv S\psi'$. Let ϕ' be ψ', which is constructed in the required way.

(3) Let ϕ be ψ_1 & ψ_2. By the induction hypothesis, there are predicates ψ_1' and ψ_2', constructed using only &, v, A, S, such that $S\psi_1 \equiv S\psi_1'$ and $S\psi_2 \equiv S\psi_2'$. Since $S\phi \equiv S\psi_1$ & $S\psi_2$, $S\phi \equiv S\psi_1'$ & $S\psi_2'$ and $S\phi \equiv S(\psi_1'$ & $\psi_2')$. Let ϕ' be ψ_1' & ψ_2', which is constructed in the required way.

(4) Let ϕ be ψ_1 v ψ_2. By the induction hypothesis, there are predicates ψ_1' and ψ_2', constructed using only &, v, A, S, such that $S\psi_1 \equiv S\psi_1'$ and $S\psi_2 \equiv S\psi_2'$. Since $S\phi \equiv S\psi_1$ v $S\psi_2$, $S\phi \equiv S\psi_1'$ v $S\psi_2'$ and $S\phi \equiv S(\psi_1'$ v $\psi_2')$. Let ϕ' be ψ_1' v ψ_2', which is constructed in the required way.

(5) Let ϕ be $T\psi$. Since $S\phi \equiv ST\psi$, $S\phi$ is true and so ϕ' can be any significant atomic wff of the class theory.

(6) Let ϕ be $(Ax')\Psi(x')$. By the induction hypothesis, there is a predicate $\Psi'(x')$, constructed using only &, v, A, S, such that $S\Psi(x') \equiv S\Psi'(x')$. Since $S\phi \equiv (Ax')S\Psi(x')$, $S\phi \equiv (Ax')S\Psi'(x')$ and $S\phi \equiv S(Ax')\Psi'(x')$. Let ϕ' be $(Ax')\Psi'(x')$, which is constructed in the required way.

(7) Let ϕ be $(Sx')\Psi(x')$. By the induction hypothesis, there is a predicate $\Psi'(x')$, constructed using only &, v, A, S, such that $S\Psi(x') \equiv S\Psi'(x')$. Since $S\phi \equiv (Sx')S\Psi(x')$, $S\phi \equiv (Sx')S\Psi'(x')$ and $S\phi \equiv S(Sx')\Psi'(x')$. Let ϕ' be $(Sx')\Psi'(x')$, which is constructed in the required way.

(8) Let ϕ be $(Ax')(A(x') \supset \Psi(x'))$. By the induction hypothesis, there is a predicate $\Psi'(x')$, constructed using only &, v, A, S, such that $S\Psi(x') \equiv S\Psi'(x')$. Since $S\phi \equiv (Ax')(A(x') \supset S\Psi(x'))$, $S\phi \equiv (Ax')(A(x') \supset S\Psi'(x'))$ and $S\phi \equiv S(Ax')(A(x') \supset \Psi'(x'))$. Let ϕ' be $(Ax')(A(x') \supset \Psi'(x'))$, which is constructed in the required way.

(9) Let ϕ be $(Sx')(T_nA(x') \,\&\, \Psi(x'))$. By the induction hypothesis, there is a predicate $\Psi'(x')$, constructed using only &, v, A, S, such that $S\Psi(x') \equiv S\Psi'(x')$. Since $S\phi \equiv (Sx')(T_nA(x') \,\&\, S\Psi(x'))$, $S\phi \equiv (Sx')(T_nA(x') \,\&\, S\Psi'(x'))$ and $S\phi \equiv S(Sx')(T_nA(x') \,\&\, \Psi'(x'))$. Let ϕ' be $(Sx')(T_nA(x') \,\&\, \Psi'(x'))$, which is constructed in the required way.

This shows that, once one has a set of significance ranges ob-

tained from atomic predicates, then by forming all the unions and intersections of these ranges, one can obtain all the significance ranges of predicates constructed from these atomic predicates. In the case of the relations o and $\in$ of the theory in Chapter 4, the following significance ranges are obtained :
The significance range of o is the class f such that $(Az')(z'\in f \equiv (Sx')(Sy')(T(z'=\langle x',y'\rangle) \ \& \ S(x'oy')))$. Since $S(x'oy') \equiv I(x') \ \& \ I(y')$, f is the class of all ordered pairs of individuals. By taking either x' or y' as constant, the significance range of 1-tuples resulting from this will consist of all individuals, i.e. it will be the set I.
The significance range of $\in$ is the class f such that $(Az')(z'\in f \equiv (Sx')(Sy')(T(z'=\langle x',y'\rangle) \ \& \ S(x'\in y')))$. f is the class of all ordered pairs, $\langle x',y'\rangle$, such that $x'\in V$ (the class of all sets and individuals) and $y'\in$ the class of all sets (call it T.). If x' is taken as a constant then the significance range of 1-tuples resulting from this will be the class of all sets. If y' is taken as a constant then the significance range of 1-tuples resulting from this will be the class of all sets and individuals.

If one considers the atomic predicate $x'\in k_0$, for some particular individual k_0, then its significance range is the null class, $\emptyset$. Thus the atomic predicates formed using o and $\in$ yield the four significance ranges consisting of 1-tuples, $\emptyset$, I, T and V. Notice here the unusual situation where one significance range is a difference of two others, i.e. T=V-I or I=V-T. This is brought about by the rather technical use of the word 'overlaps' where it is applied to all individuals.

It is clear that all significance ranges of predicates, formed using only the relations o and $\in$, are of the form, $x_1 \times \ldots \times x_\ell$, where x_i is either I, T or V, for all i, or are empty. As stated at the end of Chapter 4, ordinary language predicates can also be added to the relations o and $\in$ to form predicates which have significance ranges. So the significance ranges formed above can be enhanced by the use of ordinary language predicates.

(ii) Homogeneous and Heterogeneous Relations.

We will now look into relations in general, but we will first consider 2-place relations. Goddard, in [9]. p.153-162, distinguishes two types of relations : homogeneous and heterogeneous. He defines the significant domain, call it D_R, of the relation R such that $(Az')(z'\in D_R \equiv S(Sx')(z'Rx'))$ holds for D_R. He also defines the significant converse domain, call it C_R, of the relation R such that $(Az')(z'\in C_R \equiv S(Sx')(x'Rz'))$ holds for C_R. He defines R as homogeneous if x'Ry' is significant for an arbitrary choice of x' from D_R and y' from C_R, i.e. $S(x'Ry') \equiv S(Sy')(x'Ry')$ & $S(Sx')(x'Ry')$. If f is the significance range of R, i.e. (Az') $(z'\in f \equiv (Sx')(Sy')(T(z'=\langle x',y'\rangle) \;\&\; S(x'Ry')))$, then $\langle x',y'\rangle\in f \equiv$ $S(Sy')(x'Ry')$ & $S(Sx')(x'Ry')$. The domain of f, $\mathfrak{D}(f)$, is defined by : $(Ax')(x'\in \mathfrak{D}(f) \equiv (Sy')(\langle x',y'\rangle\in f))$. Hence $x'\in \mathfrak{D}(f) \equiv (Sy')$

$S(x'Ry')$ and $x' \in \mathfrak{D}(f) \equiv S(Sy')(x'Ry')$. The range, $\mathfrak{R}(f)$, is defined by : $(Ay')(y' \in \mathfrak{R}(f) \equiv (Sx')(\langle x',y' \rangle \in f))$. Hence $y' \in \mathfrak{R}(f) \equiv (Sx')$ $S(x'Ry')$ and $y' \in \mathfrak{R}(f) \equiv S(Sx')(x'Ry')$. By the homogeneity of R, $\langle x',y' \rangle \in f \equiv x' \in \mathfrak{D}(f)$ & $y' \in \mathfrak{R}(f)$, and hence $f = \mathfrak{D}(f) \times \mathfrak{R}(f)$. An alternative definition of homogeneity is that R is homogeneous iff its significance range f satisfies the identity, $f = \mathfrak{D}(f) \times \mathfrak{R}(f)$. In fact the domain, $\mathfrak{D}(f)$, and the range, $\mathfrak{R}(f)$, of the significance range f of the relation R are the same as the significant domain, D_R, of R and the significant converse domain, C_R, of R.

Goddard defines R as <u>heterogeneous</u> if R is not homogeneous. He says that he introduces them merely to exclude them from the more interesting homogeneous relations. He introduces four types of homogeneous relation. (i) R is a <u>family relation</u> if $D_R = C_R$. (ii) R is a <u>nest relation</u> if $D_R \subset C_R$ or $C_R \subset D_R$. (iii) R is a <u>categorial relation</u> if $D_R \cap C_R = \emptyset$. (iv) R is a <u>partial categorial relation</u> if $D_R \cap C_R \neq \emptyset$, $D_R \not\subseteq C_R$ and $C_R \not\subseteq D_R$. He then discusses each one in turn giving examples of each type. However, in the case of the partially categorial relation, he gives the examples of membership and ownership, which I think are dubious. He says, on p.162, "it is never significant to say that a class belongs to an individual, a set or a class." According to the notion of class I have given this is certainly not so. Also according to an intuitive notion of class this is not so. But he has presented a special class theory along the lines of Bernays' theory. So the notion of class here is really a technical one and technical notions can be invented to serve some formal purpose without necessarily corresponding exactly to an intuitive notion. This, I suppose, is similar to Goodman's notion of overlapping of individuals, which produces a significance range which is the difference of two significance ranges. Goddard, however, admits in the case of the relation of ownership that it is not very plausible because people can own people. I will come back to the question of partial categorial relations later in the chapter.

We will now move on to consider heterogeneous relations. Let f be the significance range of the relation R. If R is heterogeneous then there may be a proper subclass g of f such that $g = \mathcal{D}(g) \times \mathcal{R}(g)$. In this case, R could be called <u>partially homogeneous</u> because there is some part of its significant domain D_R and some part of its significant converse domain C_R from which arbitrary choices, x' and y', may be made so that x'Ry' is significant. But this can be done in general with any heterogeneous relation. Let $x_0' \in \mathcal{D}(f)$. Form the class G of all y' such that $S(x_0'Ry')$. Now form the class H of all x' such that $(Ay')(S(x_0'Ry') \equiv S(x'Ry'))$. Then the class, $g_{x_0'}$, which is the Cartesian product of G and H, is a non-empty proper subclass of f and $g_{x_0'} = \mathcal{D}(g_{x_0'}) \times \mathcal{R}(g_{x_0'})$. One can also form a similar class $g_{y_0'}$ by choosing a set y_0' from $\mathcal{R}(f)$. Thus each member of $\mathcal{D}(f)$ generates a non-empty class g such that $g \subset f$ and $g = \mathcal{D}(g) \times \mathcal{R}(g)$ and also each member of $\mathcal{R}(f)$ generates such a class g. Hence $f = \bigcup_{x_0' \in \mathfrak{D}(f)} g_{x_0'}$, where $x_0' \in \mathfrak{D}(g_{x_0'})$ and $g_{x_0'} = \mathfrak{D}(g_{x_0'}) \times \mathfrak{R}(g_{x_0'})$, and also $f = \bigcup_{y_0' \in \mathfrak{R}(f)} g_{y_0'}$, where $y_0' \in \mathfrak{R}(g_{y_0'})$ and $g_{y_0'} = \mathfrak{D}(g_{y_0'}) \times \mathfrak{R}(g_{y_0'})$. In general, $f = \bigcup_{h \in j} g_h$, where $g_h = \mathfrak{D}(g_h) \times \mathfrak{R}(g_h)$, for some index class j. (The g_h's are distinct.)

Let $f = \bigcup_{x_0' \in \mathfrak{D}(f)} g_{x_0'}$, where $x_0' \in \mathfrak{D}(g_{x_0'})$ and $g_{x_0'} = \mathfrak{D}(g_{x_0'}) \times \mathfrak{R}(g_{x_0'})$, as constructed above. Let g be one of the $g_{x_0'}$'s, let $x' \in \mathfrak{D}(g)$ and let $y' \in \mathfrak{R}(f) - \mathfrak{R}(g)$, assuming that $\mathfrak{R}(g) \neq \mathfrak{R}(f)$. Since $x' \in \mathfrak{D}(g)$, by the method of construction, $(Ay')(S(x_0'Ry') \equiv S(x'Ry'))$, for some $x_0' \in \mathfrak{D}(g)$. By the method of construction of $\mathfrak{R}(g)$, $y' \in \mathfrak{R}(g) \equiv S(x_0'Ry')$, and hence $y' \in \mathfrak{R}(g) \equiv S(x'Ry')$. Since $y' \notin \mathfrak{R}(g)$, $\sim S(x'Ry')$. Hence, for every $x' \in \mathfrak{D}(g)$, $\mathfrak{R}(g)$ is the significance range of $x'Ry'$.

However, if $x' \in \mathfrak{D}(f) - \mathfrak{D}(g)$ and $y' \in \mathfrak{R}(g)$, assuming that $\mathfrak{D}(f) \neq \mathfrak{D}(g)$, then there is nothing in the construction to prevent $x'Ry'$ from being significant.

Let $\mathfrak{D}(g_{x_0'}) \cap \mathfrak{D}(g_{x_1'}) \neq \emptyset$. Let $x' \in \mathfrak{D}(g_{x_0'}) \cap \mathfrak{D}(g_{x_1'})$. By the method of construction of the $g_{x_i'}$'s, $S(x'Ry') \equiv S(x_0'Ry')$ and $S(x'Ry') \equiv S(x_1'Ry')$. Hence $S(x_0'Ry') \equiv S(x_1'Ry')$ and $y' \in \mathfrak{R}(g_{x_0'}) \equiv y' \in \mathfrak{R}(g_{x_1'})$. Hence $\mathfrak{R}(g_{x_0'}) = \mathfrak{R}(g_{x_1'})$ and the class of all x' such that $(Ay')(S(x_0'Ry') \equiv S(x'Ry'))$ is identical with the class of all x' such that $(Ay')(S(x_1'Ry') \equiv S(x'Ry'))$. Hence $\mathfrak{D}(g_{x_0'}) = \mathfrak{D}(g_{x_1'})$. This shows that if $\mathfrak{D}(g_{x_0'}) \cap \mathfrak{D}(g_{x_1'}) \neq \emptyset$ then $\mathfrak{D}(g_{x_0'}) = \mathfrak{D}(g_{x_1'})$

and $g_{x_0'}=g_{x_1'}$. Hence, if one chooses a class of members x_i' of $\mathfrak{D}(f)$ such that all the classes $g_{x_i'}$ are distinct and if x_i' and x_j' are two such members, then $\mathfrak{D}(g_{x_i'}) \cap \mathfrak{D}(g_{x_j'})=\emptyset$. Hence $g_{x_i'} \cap g_{x_j'}=\emptyset$ and f is a disjoint union of these $g_{x_i'}$'s.

However, the $g_{x_i'}$'s do not necessarily determine homogeneous relations with $\mathfrak{D}(g_{x_i'})$ as significant domain and $\mathfrak{R}(g_{x_i'})$ as significant converse domain because, although arbitrary choice from these domains yields significance and arbitrary choice from $\mathfrak{D}(g_{x_i'})$ and from the complement of $\mathfrak{R}(g_{x_i'})$ yields non-significance, arbitrary choice from the complement of $\mathfrak{D}(g_{x_i'})$ and from $\mathfrak{R}(g_{x_i'})$ does not necessarily yield non-significance.

If $f=\bigcup_{y_0' \in \mathfrak{R}(f)} g_{y_0'}$, where $y_0' \in \mathfrak{R}(g_{y_0'})$ and $g_{y_0'}=\mathfrak{D}(g_{y_0'})\times \mathfrak{R}(g_{y_0'})$, then similar results to those shown above will hold with the domains and the ranges reversed. Hence, if one chooses a class of members y_i' of $\mathfrak{R}(f)$ such that all the classes $g_{y_i'}$ are distinct and if y_i' and y_j' are two such members, then $\mathfrak{R}(g_{y_i'}) \cap \mathfrak{R}(g_{y_j'})=\emptyset$. Hence $g_{y_i'} \cap g_{y_j'}=\emptyset$ and f is a disjoint union of these $g_{y_i'}$'s.

Let $f=\bigcup_{h \in j} g_h$, where $g_h=\mathfrak{D}(g_h)\times\mathfrak{R}(g_h)$, for some index class j. Let the g_h's be distinct and let $g_h \cap g_{h'}=\emptyset$, for all h, h'$\in$j. g_h's can be chosen so that $\mathfrak{D}(g_h) \cap \mathfrak{D}(g_{h'})=\emptyset$, for all h, h'$\in$j. Also g_h's can be chosen so that $\mathfrak{R}(g_h) \cap \mathfrak{R}(g_{h'})=\emptyset$, for all h, h'$\in$j. It would be interesting to consider the consequences of being able to choose g_h's so that $\mathfrak{D}(g_h) \cap \mathfrak{D}(g_{h'})=\emptyset$ and $\mathfrak{R}(g_h) \cap \mathfrak{R}(g_{h'})=\emptyset$, for all h, $h' \in j$. Such a relation R will be called a <u>stratified hetero-geneous relation.</u>[1] This means that the significant domain D_R and the significant converse domain C_R can be divided up into an equal number of disjoint subclasses. One subclass of D_R will correspond with one subclass of C_R such that by making an arbitrary choice, x' from the subclass of D_R and y' from the subclass of C_R, $x'Ry'$ will be significant. However, if one chooses an x' from a subclass of D_R and a y' from a subclass of C_R, which does not correspond with the subclass of D_R from which x' is chosen, then $x'Ry'$ will be non-significant.

To show this, let $x'Ry'$ be significant. Let $x' \in \mathfrak{D}(g_h)$ and let $y' \in \mathfrak{R}(g_{h'})$, where $h \neq h'$ and $h, h' \in j$. Hence $\langle x', y' \rangle \in f$ and $\langle x', y' \rangle \in g_k$, for some $k \in j$. Also $x' \in \mathfrak{D}(g_k)$ and $y' \in \mathfrak{R}(g_k)$. Since $h \neq h'$, $k \neq h$ or $k \neq h'$. If $k \neq h$, then by the definition of stratified heterogeneity, $\mathfrak{D}(g_k) \wedge \mathfrak{D}(g_h) = \emptyset$, contradicting $x' \in \mathfrak{D}(g_k)$ and $x' \in \mathfrak{D}(g_h)$. If $k \neq h'$, then by the definition of stratified heterogeneity, $\mathfrak{R}(g_k) \wedge \mathfrak{R}(g_{h'}) = \emptyset$, contradicting $y' \in \mathfrak{R}(g_k)$ and $y' \in \mathfrak{R}(g_{h'})$. Hence $x'Ry'$ is non-significant.

Hence a stratified heterogeneous relation R is a disjunction[2]

---

(1) : These relations are mentioned in [10], p.255 and in note 15, in the context of zeugmas.

(2) : This is not a classical disjunction but the disjunction, v. See note 15, p.264, [10].

of a class of homogeneous relations R_h such that, for any $x' \in D_R$ and $y' \in C_R$ such that $S(x'Ry')$, there is a unique relation R_h such that $S(x'R_hy')$. The relations R_h are homogeneous because they have a significance range g_h such that $\mathfrak{D}(g_h)$ is the significant domain, $\mathfrak{R}(g_h)$ is the significant converse domain and $g_h = \mathfrak{D}(g_h) \times \mathfrak{R}(g_h)$. The relation R_h is unique because the class of ordered pairs $\langle x', y' \rangle$ such that $x'Ry'$ is true is determined as the intersection of g_h with the class of ordered pairs $\langle x', y' \rangle$ such that $x'Ry'$ is true.

Hence the relation R is ambiguous between the relations R_h. This is typified by the example of the relation $\in$ under Type Theory. Let f be the significance range of this relation $\in$. Then $\mathfrak{D}(f)$ consists of individuals(type 0) and classes (of all types) and $\mathfrak{R}(f)$ consists of classes (of all types). The significant domain and significant converse domain are "stratified" as follows : Individuals of type 0 in $\mathfrak{D}(f)$ correspond with classes of type 1 in $\mathfrak{R}(f)$. For all n, classes of type n in $\mathfrak{D}(f)$ correspond with classes of type n+1 in $\mathfrak{R}(f)$. Each correspondence between these subclasses is such that an arbitrary choice from them yields significance and if one chooses from subclasses which do not correspond then non-significance results. Here the classes $\mathfrak{D}(f)$, $\mathfrak{R}(f)$, etc. are metatheoretic as they cannot be formulated within Type Theory itself. The relation $\in$ is ambiguous between the relations $\in_n$ (n=0, 1, 2, etc.) where $D_{\in_n}$ consists of classes or individuals of type n and $C_{\in_n}$ consists of classes of type n+1. Instead of using the general relation $\in$, one could use the relation $\in_n$ in its place. Similarly with the relation R, one could use the relations R_h in in its place. Hence any stratified heterogeneous relation can be replaced by a class of homogeneous relations.

The question now arises as to whether there are any heterogeneous relations which are not stratified heterogeneous relations. I do not think that there are any that can be obtained from atomic relations that occur in ordinary discourse. The only such relations I can think of are either homogeneous, like 'admires', 'names', 'is father of', 'owns' and 'is a member of'[1], or s~~ma~~tratified heterogeneous, like 'between', 'is next to', and other ambiguous relations. However, by using connectives and quantifiers, one may be able to artificially construct a heterogeneous relation which is not stratified.

We will now consider n-place relations as a generalisation of the 2-place relations already considered. Let f be the class of n-tuples which forms the significance range of an n-place relation R. Let $\mathfrak{D}_i(f)$ be defined as follows : $(Ax')(x' \in \mathfrak{D}_i(f) \equiv (Sy_1', \dots, y_{i-1}', y_{i+1}', \dots, y_n')SR(y_1', \dots, y_{i-1}', x', y_{i+1}', \dots, y_n))$. R is

---

(1) : 'is a member of' is taken in the ordinary discourse sense where its significant domain consists of individuals and classes and its significant converse domain consists of classes only.

homogeneous if $f=\mathcal{D}_1(f)\times\ldots\times\mathcal{D}_n(f)$ and R is heterogeneous if it is not homogeneous.

One can define generalisations of the family, nest, categorial and partial categorial relations, although it is fairly arbitrary how one does this. I will do it as follows : A homogeneous relation R is a family relation if $\mathcal{D}_1(f)=\ldots=\mathcal{D}_n(f)$, is a nest relation if $\mathcal{D}_i(f)\subseteq\mathcal{D}_j(f)$ or $\mathcal{D}_j(f)\subseteq\mathcal{D}_i(f)$ for all i, j=1,...,n and $\mathcal{D}_i(f)\subset\mathcal{D}_j(f)$ for some i,j=1,...,n, is a categorial relation if $\mathcal{D}_i(f)\cap\mathcal{D}_j(f)=\emptyset$ for all i,j=1,...,n, and is a partial categorial relation otherwise.

Let R be heterogeneous with $a_1\in\mathcal{D}_1(f)$. By the definition of $\mathcal{D}_1(f)$, there are sets or individuals $a_2,\ldots,a_n$, such that $SR(a_1, a_2,\ldots,a_n)$. Hence $a_2\in\mathcal{D}_2(f),\ldots, a_n\in\mathcal{D}_n(f)$. Form the class $h_n=\{x_n' : SR(a_1,\ldots,a_{n-1}, x_n')\}$, and hence $a_n\in h_n$. Given h_n, we can now define $h_{n-1}=\{x_{n-1}' : (Ax_n')(x_n'\in h_n \supset SR(a_1,\ldots,a_{n-2}, x_{n-1}', x_n'))\}$. $a_{n-1}\in h_{n-1}$ since $(Ax_n')(x_n'\in h_n \equiv SR(a_1,\ldots,a_{n-1}, x_n'))$. Given h_n and h_{n-1}, we can form $h_{n-2}=\{x_{n-2}' : (Ax_{n-1}')(Ax_n')(x_{n-1}'\in h_{n-1} \,\&\, x_n'\in h_n \supset SR(a_1,\ldots,a_{n-3}, x_{n-2}', x_{n-1}', x_n'))\}$. $a_{n-2}\in h_{n-2}$ since $(Ax_{n-1}')(x_{n-1}'\in h_{n-1} \equiv (Ax_n')(x_n'\in h_n \supset SR(a_1,\ldots, a_{n-2}, x_{n-1}', x_n')))$. By induction, we can form $h_1=\{x_1' : (Ax_2')\ldots(Ax_n')(x_2'\in h_2 \,\&\ldots\&\, x_n'\in h_n \supset SR(x_1',\ldots,x_n'))\}$. $a_1\in h_1$ since, by the definition of h_2, $(Ax_2')(x_2'\in h_2 \equiv (Ax_3')\ldots(Ax_n')(x_3'\in h_3 \,\&\ldots\&\, x_n'\in h_n \supset SR(a_1, x_2',\ldots,x_n')))$. Hence, non-empty classes $h_1,\ldots,h_n$ can be formed so that $(Ax_1',\ldots,x_n')(x_1'\in h_1 \,\&\ldots\&$

$x_n' \in h_n \supset SR(x_1', \ldots, x_n'))$ for each set of members $a_1, \ldots, a_{n-1}$, of $\mathfrak{D}_1(f), \ldots, \mathfrak{D}_{n-1}(f)$, respectively. Let $h_1 x \ldots x h_n = g$. Let $\mathfrak{D}_i(g)$ be defined by : $(Ax')(x' \in \mathfrak{D}_i(g) \equiv (Sy_1', \ldots, y_{i-1}', y_{i+1}', \ldots, y_n')(\langle y_1', \ldots, y_{i-1}', x', y_{i+1}', \ldots, y_n' \rangle \in g))$. Then $h_1 = \mathfrak{D}_1(g)$, $\ldots$, $h_n = \mathfrak{D}_n(g)$ and $g = \mathfrak{D}_1(g) x \ldots x \mathfrak{D}_n(g)$. Hence $f = {}_{h \in j}U\, g_h$, where $g_h = \mathfrak{D}_1(g_h) x \ldots x \mathfrak{D}_n(g_h)$ and where the g_h's are distinct, for all $h \in j$. If a relation R is such that it has a significance range $f = {}_{h \in j}U\, g_h$, where $g_h = \mathfrak{D}_1(g_h) x \ldots x \mathfrak{D}_n(g_h)$ for all $h \in j$, where the g_h's are distinct for all $h \in j$, and where $\mathfrak{D}_1(g_h) \wedge \mathfrak{D}_1(g_{h'}) = \emptyset, \ldots$, $\mathfrak{D}_n(g_h) \wedge \mathfrak{D}_n(g_{h'}) = \emptyset$, for all $h, h' \in j$, then such a relation is called a <u>stratified heterogeneous relation</u>. This means that its domains $\mathfrak{D}_1(f), \ldots, \mathfrak{D}_n(f)$ can be divided up into an equal number of disjoint subclasses. There is a one-one correspondence between the subclasses of these domains such that by making an arbitrary choice $x_1', \ldots, x_n'$ from these corresponding subclasses $R(x_1', \ldots, x_m')$ will be significant. However, if one makes a choice $x_1', \ldots, x_n'$ from subclasses that do not correspond then $R(x_1', \ldots, x_n')$ will be non-significant.

To show this, let $R(x_1', \ldots, x_n')$ be significant. Then $\langle x_1', \ldots, x_n' \rangle \in g_h$, for some $h \in j$, and $x_1' \in \mathfrak{D}_1(g_h), \ldots, x_n' \in \mathfrak{D}_n(g_h)$. Hence x_1', $\ldots$, x_n' all belong to corresponding subclasses of $\mathfrak{D}_1(f), \ldots$, $\mathfrak{D}_n(f)$, respectively.

As before a stratified heterogeneous relation R is a disjunction of a class of homogeneous relations R_h. If $SR(x_1', \ldots, x_n')$ then there is a unique relation R_h such that $SR_h(x_1',\ldots,x_n')$. The relation R is ambiguous between the relations R_h.

(iii) Atomic Significance Ranges.

In the first section of this chapter, it was stated that significance ranges, that is, if they are classes of 1-tuples, are characterised as being classes whose members exhaust one sort of thing or exhaust some sorts of things. In this section, I want to define the notion of atomic significance range so that it is characterised as a class whose members exhaust just one sort of thing and it consists of 1-tuples only. It is clear that the union of at least two mutually disjoint significance ranges cannot be an atomic significance range. So, let us define an atomic significance range as a non-empty significance range which cannot be expressed as the union of at least two mutually disjoint significance ranges. This seems to satisfy the above characterisation.

The important thing to note about the definition of an atomic significance range is that it requires a knowledge of all significance ranges that are subclasses of a given significance range in order to determine whether that given significance range is atomic or not. Hence it depends on the different predicates in a language as to whether a significance range is atomic or not.

With technical languages, where all the predicates are well-defined, it is usually an easy task to determine which are the atomic significance ranges. For example, in the theory of classes and individuals of Chapter 4, if no other predicates are added but the original ones, $\in$ and o, then the atomic significance ranges are the set of all individuals, I, the class of all sets, T, and the class of all sets and individuals, V. All of the other significance ranges of this theory are n-tuples of unions of these three, plus the null class.

However, if one takes into account the predicates of ordinary discourse, one is not always sure of a significance range being atomic. For example, whether the significance range of material objects is atomic or not depends on whether there are significance ranges of animate and of inanimate objects, say, or whether there are significance ranges of certain types of animate objects which exhaust all of the animate objects and there are significance ranges of certain types of inanimate objects which exhaust all of the inanimate objects. So it depends on the types of predicates one has in the language.

It is possible by using certain predicates to get atomic significance ranges which are quite small. For example, 'is a good creamer'[1] has a significance range of milk-giving animals. This is a strange significance range produced by using a fairly ordinary predicate but it is a bit technical. ~~However, consider the legal predicate, 'is State-assisted', say, which has as its significance~~

(1) : This example is due to Mr. R. Hughes.

range a certain race of people living in a certain country with a legal system in which the predicate is used in such a way as to apply only to this race. This is an artificial predicate determined by an arbitrary legal system of a particular country. It is possible to use such legal predicates to obtain atomic significance ranges which one would hardly want to call "a sort of thing", but these significance ranges would still be the union of the true and the false ranges of these predicates.

Next we will prove that there are only finitely many significance ranges.[1] The only way one can construct a significance range with infinitely many members is to use idealised concepts such as occur in Mathematics because it is Mathematics which deals with the infinite and contains concepts, such as sets, classes, points, lines, numbers, etc., of which there are infinitely many. Because Mathematical theory is well-defined, it is possible to count all the significance ranges that occur in it. For example, 'is prime' and 'is divisible by 2' determine the significance range of all natural numbers. 'is greater than 2.56' and 'is between 2 and 2.4' determine the significance range of all real numbers. 'has four subgroups' determines the significance range of groups. Although conventional Mathematics does not contain significance ranges, they can be introduced as indicated in the above examples. Hence there are only finitely many such significance ranges with

(1) : Here and throughout this section I will only be concerned with significance ranges of 1-tuples.

infinitely many members.

The idealised concept of type-sentence also yields infinite significance ranges. But here again, because of well-defined grammatical rules, there are only finitely many significance ranges. In general, idealised concepts are sufficiently well-defined so that one can count significance ranges and be sure that there are only finitely many.

The class of all things which are *not* idealised concepts is finite and hence there can only be a finite number of significance ranges with these things as members. Hence if one forms a significance range x containing infinitely many members it contains a subclass y which is an infinite significance range such that the class x-y is finite, such that y contains idealised concepts only and such that x-y does not contain idealised concepts. There are only finitely many significance ranges like y and only finitely many classes like x-y and hence there are only finitely many infinite significance ranges like x. Hence there are only finitely many significance ranges. This proof is subject to the assumption that all idealised concepts are sufficiently well-defined and the predicates used to describe them are fully determined.

This seems to be a fair assumption to make and it would only be refuted in a highly artificial way, as in the Theory of Types. As shown before, there are denumerably many significance ranges : individuals (type 0), classes of type 1, classes of type 2, etc.

In fact, in some recent class theories, there are a transfinite number of types and hence a transfinite number of significance ranges. Firstly, these theories are artificial and do not represent the ordinary discourse notions of membership and class. Secondly, non-significance in these theories are excluded by the formation rules or just replaced by falsity.

Using the above result, we will prove : All non-empty significance ranges are unions of mutually disjoint atomic significance ranges.

<u>Proof</u>. Let x be a non-empty significance range. If x is atomic, we are done. If x is not atomic then, by the definition of an atomic significance range, it can be expressed as the union of at least two mutually disjoint significance ranges. Let these significance ranges be x_i, where $i \in j$, an index class. $x_i \cap x_{i'} = \emptyset$, for all $i, i' \in j$. If the x_i's are all atomic, we are done. If some are not atomic then express each non-atomic significance range as the union of mutually disjoint significance ranges, and let the step by step process of forming mutually disjoint significance ranges continue. Because there are only finitely many significance ranges, there are only finitely many steps in this process and it will terminate with the formation of a union of mutually disjoint significance ranges, as required.

We will now consider the types of predicates which can be used to determine an atomic significance range, assuming we have some idea of the types of predicates used in the language. One would suspect that there is some connection between atomic significance ranges and atomic predicates. There are atomic predicates yielding non-atomic significance ranges. For example, the predicate 'owns a car'[1] has a significance range consisting of people and companies. This occurs because 'owns' is a legal term and can only legally apply to people and companies. It is clear that people form a significance range because of predicates like 'works in the library'. Also companies form a significance range because of predicates like 'merged with Consolidated'.

There are compound predicates yielding atomic significance ranges. For example, 'is blue or is hard' has a significance range consisting of extended things which is obtained by forming the union of the significance ranges of extended things and of material objects. The significance range of extended things seems to be atomic because there seems to be no predicate with all non-material extended things as a significance range. I am subject to correction on this point, but I doubt whether there are any predicates of the above type from ordinary discourse. Another example of a compound predicate with an atomic significance range is the predicate 'merged with Consolidated and owns ten cars'. Here the significance range is that of companies and this is a more clear cut atomic significance

(1) : The relation 'owns' is discussed in [9], p.162.

range than the above one.

However, I think that given an atomic significance range there is an atomic predicate with this significance range. Since an atomic significance range consists of just one sort of thing, according to the interpretation, then if it is distinguished as a significance range at all by means of predicates, there should be some atomic predicate which applies to things of this sort. The example given above, 'is blue or is hard' bears this out as 'is blue' has the same significance range as 'is blue or is hard'.

At this point, I would like to distinguish between atomic significance ranges and categories. As stated above, atomic significance ranges depend on the types of predicates one has in a language or in the system being formalised. In the literature, 'category' and 'significance range' are often used interchangeably. However, I would like to make a useful distinction by defining a category as a sort of thing, only instead of being determined by a predicate in a language or in a formal system, it is independent of language. Categories are metaphysical as opposed to significance ranges which are logical. Categories are sorts of things which the "things" of the world naturally fall into. Categories depend on the way the world is divided up.

However, they are more or less the same as atomic significance ranges. Most ordinary discourse atomic predicates will determine a category but artificial and technical predicates would have to be omitted. So, categories are formed from predicates of ordinary discourse with a certain amount of trimming down of these predicates so that categories are not made to depend on special words or phrases with artificially limited application or on predicates imposing, by mere choice of words, conditions on the subject which no category can satisfy.

The notion of category needs further elucidation, much more than can be done in this thesis, but I just wanted to draw the above distinction to avoid the conflation of these two notions, that sometimes occurs in the literature.

(iv) Sommers' Principle.

It is clear from the foregoing examples that two atomic significance ranges can be disjoint and that one can be contained in the other. But there have not been any examples where two atomic significance ranges properly intersect. Sommers, in [28] and [29], develops a theory of significance ranges in a rather different way to that in this thesis. In his theory he affirms the equivalent of 'Two atomic significance ranges do not intersect'. The two significance ranges are atomic because he "locates" a significance range in different places if the predicate determining it is ambiguous. For example, on p.177 of [28], he gives 'reasonable' two locations, one according to the use of 'reasonable' in 'A man is reasonable' and the other according to the use of 'reasonable' in 'An argument is reasonable'.

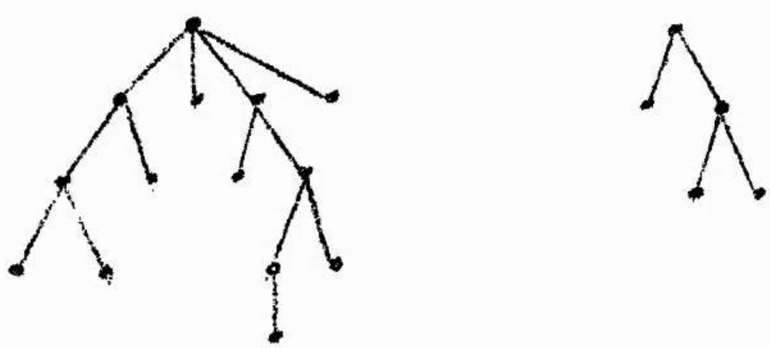
Let us examine what the proper intersection of two atomic significance ranges, x and y, entails. The intersection $x \cap y$ itself is a significance range, not necessarily atomic. It consists of some sorts of things and is properly contained in two distinct atomic significance ranges both consisting of just one sort of thing. In ordinary discourse, when some sorts of things are all of the one sort of thing, that is, they receive a more general classification, then any other more general classification is more general or less general than the first. That is, there is a total ordering of classifications of things which are members of some significance range containing some sorts of things.

Consider the example of some sorts of things which are all material objects. The significance range could be determined by the disjunction of the predicates ~~'was felled yesterday'~~ 'is a member of the British Cabinet', with the significance range of ~~trees~~ people, and 'is a good creamer', with the significance range of milk-producing animals. The members of the significance range z can be generally classified as material objects, extended things, substances (as in Aristotle's theory of substance) and individuals. Some of these classifications may be atomic significance ranges but anyway each one in the sequence contains all earlier ones in the sequence. This is an example of the total ordering of classifications.

There is another example of this in Mathematics. Consider the sequence of classifications, natural numbers, rational numbers, algebraic numbers, real numbers, complex numbers, and mathematical ~~concepts~~ objects. Here again, all natural numbers are rational numbers, all rational numbers are algebraic numbers, all algebraic numbers are real numbers, all real numbers are complex numbers, all complex numbers are mathematical ~~concepts~~ objects.

The above argument is in terms of sorts of things, but this would be borne out in the case of atomic significance ranges as determined by predicates, provided the predicates were not artificial or too technical. The only counter-examples of this total ordering of atomic significance ranges could occur by using artificial or technical predicates. For example, there could be some predicate used solely in the context of cows and bulls, so that its significance range consists just of cows and bulls. But the predicate 'is a good creamer' has a significance range of milk-producing animals. Provided there are no predicates determining a significance range of bulls or a significance range of milk-producing non-cows, the above two significance ranges are probably atomic. Let us assume that these significance ranges are atomic. Then they have a proper intersection consisting of cows. That is, cows are classified according to two different properties that they have, one of being of the species of cow and bull and the other of being milk-producing. While these two different types of classification are all right when dealing with classes, they are peculiar when dealing with significance ranges as one does not expect these significance ranges to be significance ranges at all. Essentially, they are determined by technical predicates applying to a narrow range of things. However, there are no proper intersections of categories, since significance ranges like the examples above are too small to be categories.

Anyway, I would like to assume that there are no proper intersections of atomic significance ranges. This is what I will call Sommers' Principle. It is a principle restricting the technical predicates in a way to avoid the proper intersection of two atomic significance ranges. As in Sommers, [28], the Principle yields a tree structure of atomic significance ranges. Two atomic significance ranges can be identical, one can be properly contained in the other, and they can be disjoint. Hence the set of atomic significance ranges containing a given one is totally ordered by the relation of proper containment. The tree structure is as shown :



The dots represent atomic significance ranges. If a dot X is higher than a dot Y and connected by a line going upwards only then the significance range represented by X properly contains the significance range represented by Y. Note that there is no null atomic significance range contained in all atomic significance ranges, also that there is no universal atomic significance range containing all atomic significance ranges. I think it is unlikely that the universal significance range is atomic because each thing in the universe can probably be classified in some way, as in the Aristotelian categories, so that it is contained in an atomic significance range which is not universal. Sommers' Principle has the effect of preventing anything of the form 

appearing in the tree.

~~By the results of the last section, non-empty significance ranges are finite unions of mutually disjoint atomic significance ranges. Also every member of a significance range is a member of some atomic significance range.~~ Sommers' Principle entails that if two atomic significance ranges, x and y, intersect then the intersection is an atomic significance range, which is either x or y. Also the union of two atomic significance ranges, x and y, is not an atomic significance range if x and y are disjoint, and is an atomic significance range, which is either x or y, if $x \subseteq y$ or $y \subseteq x$.

Two properly intersecting significance ranges, x and y, occur in the case of a partial categorial relation. At least one of these significance ranges must be non-atomic. Usually, either x-y or y-x is a significance range making either x non-atomic or y non-atomic, respectively. A counter-example to this would be a case where $x=x_1 \cup y_2$, where $x_1 \cap y_2 = \emptyset$, and $y=y_1 \cup x_2$, where $y_1 \cap x_2 = \emptyset$, and where $x_2 \subset x_1$, $y_2 \subset y_1$ and $x_1 \cap y_1 = \emptyset$. x_1, x_2, y_1 and y_2 are atomic significance ranges and of course, x and y are not. In this case, neither x-y nor y-x are significance ranges since $x-y = x_1 - x_2$ and x_1 is atomic and since $y-x=y_1-y_2$ and y_1 is atomic. The intersection of x and y is $x_2 \cup y_2$, which is the union of two disjoint atomic significance ranges. In this case also x and y are both non-atomic. It is easily seen that the atomic significance ranges x_1, x_2, y_1 and y_2 satisfy Sommers' Principle.

~~In the theory of homogeneous and heterogeneous relations, each of the significant domains of a heterogeneous relation must be non-atomic and a stratified heterogeneous relation can only be ambiguous between a finite number of homogeneous relations.~~

~~Also in the formal determination of significance ranges by means of predicates, the quantifiers A and S are not needed because they produce a significance range which is a finite union or intersection of significance ranges and this could be produced by using a finite disjunction or conjunction.~~

(v) Significance Ranges in the 4-valued Class Theory.

If one uses only the connectives and quantifiers as stated for the Axiom (A) of Abstraction then significance ranges can be formed as in the 3-valued theory, except that one cannot use ordered n-tuples. The definition is as follows : The unique F such that

$(AU)(U\in F \leftrightarrow S\phi(U, \bar{V}_1,\ldots,\bar{V}_i, \bar{x}_1,\ldots,\bar{x}_k))$, where ϕ is either a propositional constant or constructed using $\circ$, $\in$, ~, &, v, S(sig.), A, S(some), where the quantification is unrestricted using variables of type x or X or restricted to a predicate A(x) or A(X), in which every occurrence of a variable over classes and individuals is covered by S(sig.), and where A is constructed from atomic wffs using only ~, &, v, S(sig.), A, S(some), where the quantifiers A and S are unrestricted, is <u>the significance range of $\phi(U, \bar{V}_1,\ldots,\bar{V}_i, \bar{x}_1,\ldots,\bar{x}_k)$.</u>

However, if ϕ is allowed to be constructed using the connectives and quantifiers of standard wffs, then anomalous results can follow. These connectives and quantifiers can be used to form an s-n sublogic but are not necessarily "positive". For example, the following generalisation of the 3-valued N is such a connective which is not "positive".

N	
1	n
$\frac{1}{2}$	n
0	n
n	1

Since $\sim Sp \equiv SNp$, ~S-ranges can be formed with similar objections to those mentioned in the 3-valued case.

Similarly to the 3-valued case, it can be shown that if F is the significance range of ϕ, constructed as above, then there is a predicate $\emptyset'$ constructed without ~ and without S(sig.) such that F is the significance range of $\emptyset'$. Hence, once one has a set of significance ranges obtained from atomic predicates, then by forming all the unions and intersections of these ranges one can obtain all the significance ranges of predicates constructed from these atomic predicates.

The most important difference between the significance ranges of the 3-valued and 4-valued theories is that of the universal significance ranges of classes and of classes and individuals. In the 3-valued theory one can only form the significance ranges of all sets and of all sets and individuals, but in the 4-valued theory one can form the significance ranges of all classes and of all classes and individuals. The difficulty with the 3-valued theory is that it is false for proper classes to be members and yet the significance range of the predicate '$\in\bar{x}$' excludes them. However in the 4-valued theory there is no such difficulty.

The theory of homogeneous and heterogeneous relations still holds in the 4-valued theory but one must avoid the use of n-tuples and Cartesian products. That is, R is <u>homogeneous</u> if S(XRY) ≡ S(SY)(XRY) & S(SX)(XRY), for all X and Y, R is <u>heterogeneous</u> if R is not homogeneous and R is <u>stratified heterogeneous</u> if its significant domain and significant converse domain can be divided up into an equal number of disjoint subclasses so that there is a one-one correspondence between them and any choice made from corresponding subclasses yields significance while any choice from non-corresponding subclasses yields non-significance. Similarly, for n-place relations, these types of relations can be defined without the use of n-tuples or Cartesian products.

There is essentially no difference to the theory of atomic significance ranges and Sommers' Principle by using the 4-valued class theory.

(vi) Axiomatisation of Significance Range Theory.

The theory of significance ranges presented above was done informally using a formal 3 or 4-valued class theory as a background. It is also possible, however, to axiomatise the theory of significance ranges and incorporate it into a formal theory of classes. It is doubtful whether the predicate 'is a significance range' can be defined explicitly in either of the 3 or 4-valued class theories because it depends on a general abstraction-type axiom. So one must take as primitive the predicate 'is a significance range', call it R(f) (or R(F)), and use axioms about significance ranges in order to pin down the meaning of the primitive predicate.

This can be done as follows for the 3-valued theory : Add the primitive predicate, R. Add the definition of variables ranging over significance ranges :

$(As)\phi(s) =_{df} (Af)(R(f) \supset \phi(f))$.

$(Ss)\phi(s) =_{df} (Sf)(T_n R(f) \& \phi(f))$.

Let r, s, t, etc. be such variables over significance ranges.

Add the definition of atomic significance ranges :

$At(r) =_{df} \sim r=\emptyset \ \& \ \sim(Ss)(St)(r=s\cup t \ \& \ s\cap t=\emptyset)$.

Add the axioms dealing with significance ranges :

S1. $(Ss)(Az')(z'\in s \equiv (Sx_1',\ldots,x_\ell')(T(z'=\langle x_1',\ldots,x_\ell'\rangle) \ \& \ S\phi(x_1',\ldots,x_\ell', y_1,\ldots,y_m)))$, where ϕ is constructed using only the connectives ~, &, v and T and only the quantifiers, A and S(restricted and unrestricted).

S2. $At(s) \ \& \ At(t) \supset . s\cap t=\emptyset \vee s\subseteq t \vee t\subseteq s$.

S3. $SR(f) \ \& \sim SR(k)$.

The informal definition of atomic significance range is equivalent to the formal definition because one can always form the union of all but one of the disjoint significance ranges in the event of there being more than two disjoint significance ranges.

The axiom S2 is Sommers' Principle.

The axiom S3 states that the predicate 'is a significance range' has a significance range of classes.

The special class Axiom B is extended so that ϕ can be constructed using the predicate R as well as the relations o and $\in$. This then allows the formation of the class of all significance ranges (which are sets) as the unique class S such that $(Az')(z'\in S \equiv TR(z'))$.

~~Hence another axiom can be added :~~

~~S4. (Sg)(S ≃ g & g ⊆ ω).~~

~~Axiom S4 states that the class of all significance ranges (which~~

are sets) is finite. Using this one can then prove that all non-empty significance ranges are unions of disjoint atomic significance ranges.

In order to make the last axiom sensible, one must add ordinary discourse predicates to the formal system. If predicates which are artificial or too technical are added then Axioms S2 or S4 could be contradicted by the formation of significance ranges by Axiom S1.

The axiomatisation of significance range theory using the 4-valued class theory as a background is similar to that above. The variables ranging over significance ranges are R, S, T, etc. They are defined as follows using the primitive predicate F :

$(AR)\phi(R) =_{df} (AF)(R(F) \supset \phi(F))$.

$(SR)\phi(R) =_{df} (SF)(T_n R(F) \,\&\, \phi(F))$.

Atomic significance range is defined as follows :

$At(R) =_{df} (SX)T(X\ R) \,\&\, \sim(SS)(ST)((AX)(T(X \in R) \equiv T(X \in S) \vee T(X \in T)) \,\&\, \sim(SX)(T(X \in S) \,\&\, T(X \in T)))$.

The axioms are as follows :

S1. $(SR)(AX)(X \in R \equiv S\phi(X, V_1, \ldots, V_i, x_1, \ldots, x_k))$, where ϕ is constructed as in the definition of significance range in the last section.

S2. $At(S) \,\&\, At(T) \supset . \sim(SX)(T(X \in S) \,\&\, T(X \in T)) \vee (AX)(T(X \in S) \supset T(X \in T)) \vee (AX)(T(X \in T) \supset T(X \in S))$.

S3. $CR(F) \,\&\, \sim SR(k)$.

The Abstraction Axiom A is extended so that $\emptyset$ can be constructed using the predicate R as well as o and . The class of all significance ranges, S', can then be formed : $(AX)(X \in S' \leftrightarrow R(X))$.

~~But the finiteness of the class S' cannot be stated very satisfactorily in this theory. However, one can add the axiom :~~

~~S4. G(X∈R).~~

~~This allows a simplification of the definition of 'At' and of the statement of Axiom S2.~~

There are some advantages and disadvantages about these two class theories as used for the formalisation of significance range theory. The advantage of the 3-valued theory is that it allows the formation of n-tuples leading to a general notion of significance range for relations. The disadvantage of the 3-valued theory is that one cannot form the significance range of all classes but one can only form the significance range of all sets, despite the fact that it is false for proper classes to belong to classes and hence significant for proper classes to belong to classes, and also that it is significant for classes and individuals to belong to classes, i.e. to proper classes as well as sets. The advantage of the 4-valued theory is that one can form the significance range of all classes. The disadvantage of the 4-valued theory is that ~~one cannot satisfactorily state the finiteness of the number of significance ranges and~~ one cannot use n-tuples for defining significance ranges of relations.

Nevertheless it is essential for there to be some class theory which can be used as a background for the formalisation of significance range theory. On the whole, the 3-valued theory is easier to deal with but one must make allowances for the significance ranges of all sets and of all sets and individuals.

CONCLUSION.

Firstly, I will go through each of the aims and show where the results for each can be found.

The 3-valued logic with values, truth, falsity and non-significance is developed in Chapter 1, where the sentential logic is developed, and in Chapter 2, where the predicate logic is developed. The theory of significance ranges is developed in Chapter 8. Goodman's theory of individuals is developed in Chapter 3. Here there is incorporated into it a theory of sets of individuals. In Chapters 4 and 7 the theory of individuals is incorporated into a 3-valued and a 4-valued class theory, respectively. The problem of distinguishing the null class from individuals is solved in each of the three theories it appears in, i.e. in Chapters 3, 4 and 7. In Chapter 5, by using a 3-valued logic, the paradoxes of class theory are avoided as shown by the relative consistency proof. The same applies to extensions of that theory in Chapters 6 and 7. The 3- and 4-valued logics used in these theories are developed in Chapters 1 and 2. I have also shown the relative consistency of the class theory of Chapter 4, using the 3-valued significance logic.

Secondly, I will assess the 3-valued approach in solving the class paradoxes. Certainly the 3-valued logic allows a consistent extension of any 2-valued theory of classes and hence generalises any such theory. As pointed out in Chapter 5, there are many problems in the development of the 3-valued class theory but, as pointed out in Chapter 6, the theory is strong enough for the development of Mathematics because it can contain NBG as a 2-valued sub-theory. One of the problems about the 3-valued approach is that there is a restriction on the connectives and quantifiers that can be used in the Abstraction Axiom. At the end of Chapters 6 and 7 there was a characterisation given for all the connectives and quantifiers that could be used in the Abstraction Axiom so that the consistency proof given would hold. This restriction of connectives and quantifiers is essential to avoid contradictions which arise if certain other connectives and quantifiers are used. But this leaves open the question of what to do with predicates formed using connectives or quantifiers of the type that can lead to contradictions. For example, form the class $\mathcal{R}$ such that $(AX)(X \in \mathcal{R} \leftrightarrow \sim T(X \in X))$. Then $\mathcal{R} \in \mathcal{R} \leftrightarrow \sim T(\mathcal{R} \in \mathcal{R})$. Since the right hand side is 2-valued so must the left hand side be. Hence there is a contradiction. This applies to any finitely or infinitely valued logic with a 2-valued T operator. Hence many-valued logics cannot be used to solve this paradox. The reason for T's being 2-valued is that it is a meta-theoretic notion but formalised in the theory. The same applies to F, P, C, $\supset$ and $\equiv$. '$p \supset q$' is read as 'If ~~p is true~~ it is true that p then q'. $\leftrightarrow$ is not meta-theoretic as it is not wholly 2-valued. Similarly with $\rightarrow$. However, a 2-valued version of $\leftrightarrow$ would be meta-theoretic, meaning 'has the same value as'.

Anyway, $\rightarrow$ and $\leftrightarrow$ can be used in an Abstraction Axiom using Lukasiewicz infinitely-valued logic. Hence a levels of language argument can be used to determine what connectives and quantifiers lead to paradoxes using any many-valued logic. I think the general use of $\rightarrow$ and $\leftrightarrow$ should be avoided because, as mentioned in the Introduction, peculiar classes can be formed. As for "classes" like $\mathcal{R}$, as above, I do not think that these should be classes at all. $\mathcal{R}\in\mathcal{R}$ cannot be given any logical value and one is forced to say that $\mathcal{R}$ is not a class or that $\mathcal{R}\in\mathcal{R}$ is not a sentence capable of taking a value. If $\mathcal{R}$ is a class then the sentence must be capable of taking a value and so $\mathcal{R}$ is not a class.

Since classes, by their very nature, are generated by predicates, as stated in the Introduction, one must be able to give general criteria for the construction of such predicates. If certain predicates give contradictions, it seems implausible to disallow just those predicates that give contradictions. The general criteria I have given are that the connectives and quantifiers should be restricted so that only those used in forming standard wffs should be used, and that they should be restricted so that none yielding peculiar classes should be used. This reduces them to $\sim$, & and A and allows the possibility of U=V being used as a predicate as well as $U\in V$. [In fact, one of the unsolved problems in this thesis is whether U=V can be consistently added as another predicate which can be used to form predicates to generate classes.]

This gives a theory satisfying the Boolean operations. Its advantages are that one can form unrestricted complements and one can form the universal class. This is an aid to significance range theory also, as one can form the significance range of all classes in the 4-valued theory. Other advantages are elucidated at the end of Chapter 6.

I will now consider the semantic paradoxes and see whether a many-valued logic can be used to avoid these. Take the example, 'This very sentence is false'. If 'is false' is 2-valued like the F of the preceding section then there is no many-valued logic in which the above sentence can be consistently assigned a value, for similar reasons to those of the preceding section. If 'is false' just negates the sentence so that 'is false' can be many-valued, then the Lukasiewicz 3-valued logic can be used to consistently assign a value to the sentence, as the sentence takes the value $\frac{1}{2}$.

Consider now the heterologicality paradox. Here 'heterological' is heterological if and only if 'heterological' does not apply to itself, i.e. 'heterological' is not heterological. The 'not' seems to be a predicate negation as opposed to a meta-theoretic falsehood and so ' 'heterological' is heterological' can be assigned the value $\frac{1}{2}$ in a Lukasiewicz 3-valued logic. So far as semantic paradoxes are concerned, it depends on the interpretation of the negation as to whether one can use a Lukasiewicz 3-valued logic or not be able to use any many-valued logic at all.

I will now consider significance ranges and categories. There needs to be further research done into the notion of category and its comparison with that of atomic significance range. This would have to involve a study of the predicates one is prepared to reject when forming categories. One would also need to determine what predicates are safe to use ~~so that the number of atomic significance ranges is finite and~~ so that Sommers' Principle holds. ~~If the number of atomic significance ranges is finite then one should determine exactly what this finite list would be.~~

BIBLIOGRAPHY.

[1] : C. C. Chang, "The Axiom of Comprehension in Infinite-valued Logic", Math. Scand., Vol. 13 (1963), p.9-30.

[2] : A. Church, "Introduction to Mathematical Logic", Princeton, 1956.

[3] : P. J. Cohen, "Set Theory and the Continuum Hypothesis", Benjamin, New York, 1966.

[4] : T. Drange, "Type Crossings", Moulton, The Hague, 1966.

[5] : J. Fenstad, "On the Consistency of the Axiom of Comprehension in the Lukasiewicz Infinite Valued Logic", Math. Scand., Vol. 14 (1964), p.65-74.

[6] : F. Fitch, "Symbolic Logic", New York, 1952.

[7] : A. Fraenkel and Y. Bar-Hillel, "Foundations of Set Theory", North Holland, 1958.

[8] : P. C. Gilmore, "The Consistency of Partial Set Theory without Extensionality", I.B.M. Research Report, RC1973, Dec. 21st, 1967.

[9] : L. Goddard, "Predicates, Relations and Categories", Australasian Journal of Philosophy, Vol. 44 (1966), p.139-171.

[10] : L. Goddard, "Towards a Logic of Significance", Notre Dame Journal of Formal Logic, Vol. 9 (1968), p.233-264.

[11] : L. Goddard, "Nonsignificance", Australasian Journal of Philosophy, May 1970, forthcoming.

[12] : L. Goddard and R. Routley, "Significance Theory", Scottish

Academic Press, forthcoming.

[13] : N. Goodman, "The Structure of Appearance", Harvard, 1951.

[14] : L. J. Lafleur, Review of [16], Journal of Symbolic Logic, Vol. 5 (1940), p.113-4.

[15] : K. Lambert, "On the No Type Theory of Significance", Australasian Journal of Philosophy, Vol. 46 (1968), p.79-86.

[16] : Leonard and Goodman, "The Calculus of Individuals and Its Uses", Journal of Symbolic Logic, Vol. 5 (1940), p.45-55.

[17] : E. Mendelson, "Introduction to Mathematical Logic", Van Nostrand, 1964.

[18] : W. V. Quine, "Word and Object", Wiley, 1960.

[19] : W. V. Quine, "Set Theory and its Logic", Harvard, 1963.

[20] : Raphael Robinson, "The Theory of Classes, a Modification of Von Neumann's System", Journal of Symbolic Logic", Vol. 2 (1937), p.29-36.

[21] : J. Rosser and A. Turquette, "Many-valued Logics", North Holland, 1952.

[22] : R. Routley, "On a significance Theory", Australasian Journal of Philosophy, Vol. 44 (1966), p.172-209.

[23] : R. Routley, "The Need for Nonsense", Australasian Journal of Philosophy, forthcoming.

[23a] : R. Routley and V. Routley, "Categories — expressions or things?," Theoria, Vol. 35 (1969) p215-23.

[24] : G. Ryle, "Categories", reprinted in Flew (Ed.), "Logic and Language", Series II, Blackwell, 1953.

[25] : T. Skolem, "Bemerkungen zum Komprehensionsaxiom", Zeits-

chrift für Math. Logik und Grundlagen der Math., Bd. 3 (1957), p.1-17.

[26] : T. Skolem, "A Set Theory Based on a Certain 3-valued Logic", Math. Scand., Vol. 8 (1960), p.127-136.

[27] : T. Skolem, "Studies on the Axiom of Comprehension", Notre Dame Journal of Formal Logic, Vol. 4 (1963), p.162-170.

[28] : F. Sommers, "The Ordinary Language Tree", Mind, Vol. 68 (1959), p.160-185.

[29] : F. Sommers, "Types and Ontology", Phil. Rev., Vol. 72 (1963), p.327-363.

[30] : P. Suppes, "Axiomatic Set Theory", Van Nostrand, 1960.

[31] : M. Wajsberg, "Axiomatisation of the 3-valued Propositional Calculus", in "Polish Logic, 1920-1939", ed. by Storrs McCall, Oxford, 1967, p.264-284.

[32] : H. Wang, "What is an Individual", Phil. Rev., Vol. 62 (1953), p.413-420.

[33] : H. Wang, "Survey of Mathematical Logic", North Holland, 1963.

[34] : Whitehead and Russell, "Principia Mathematica", Cambridge, 1st edition, 1910.